A categorical outlook on relational modalities and simulations

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Abstract

We characterise bicategories of spans, relations and partial maps universally in terms of factorisations involving *maps*. We apply this characterisation to show that the standard modalities $\Box$ and $\Diamond$ arise canonically as extensions of a predicate logic from *functions* to (abstract) *relations*. When relations and partial maps are representable, we exhibit *logical predicates* for the power-object and partial-map-classifier monads. We also show that the $\Box$ modality gives the relevant pullbacks of subobjects in the internal logic of categories of partial maps. Organizing modal formulae fibrationally, we exhibit an intrinsic relationship between their *satisfaction* relative to transition systems and the notion of *simulation*. In this setting, we use the biclosed structure of the bicategory of relations to give a new proof of the standard fact that *observational similarity* implies *similarity*.

Key words: bicategories of relations, modal logic, simulations

1. Introduction

We set about exploring intuitionistic/constructive *modalities* from a categorical logic perspective, along the lines of Lawvere's analysis [16] of logical connectives and quantifiers as adjoint functors. We are primarily interested in the use of modal formulae to analyse properties of transition systems and we thus consider modal logic as a logic of *sets and relations*, in the same spirit as first-order predicate logic is a logic of *sets and functions*. In the first part of the paper we give a precise mathematisation of this dictum: we show that the standard interpretations in first-order predicate logic of the relational modalities **possibility** $\langle R\rangle$ and **necessity** $[R]$ arise as *canonical* extensions of a predicate logic (viewed as a fibration or pseudo-functor) over a category $\mathbf{B}$ (thought of as $\mathcal{S}et$) to its associated bicategory of relations $\mathcal{R}el(\mathbf{B})$.

Such canonical extensions are induced by a universal property of the embedding $\eta_{\mathbf{B}} : \mathbf{B} \to \mathcal{R}\mathbf{el}(\mathbf{B})$ (Theorem 2.3) which construes a function as a relation via its graph. Along the same lines, we show how the same situation with respect to the subcategory of partial maps accounts for the operation of *substitution* in the *logic of partial elements*. Continuing this analysis, we show in §3.2 that when relations/partial maps are representable, *i.e.* the embedding

$\eta_{\mathbf{B}}$ admits a right adjoint, the same its true for its counterpart at the predicate level. We thus obtain *logical predicates* formulae for the powerobject and partial-map-classifier monads, along the lines of [8].

In the second part, we switch our attention to the interpretation of modal formulae in transition systems. We present in §5 an algebraic reformulation of the notion of simulation or rather, its dual which we christen **opsimulation**. We exhibit a fundamental relationship between the satisfaction of modal formulae in a labelled transition system and the notion of opsimulation between such systems. More precisely, we show how modal formulae can be organized into a transition system $\underline{\mathcal{P}S}$ such that the satisfaction relation becomes an opsimulation between the given system and $\underline{\mathcal{P}S}$. Furthermore, this opsimulation enjoys a universal property (Theorem 6.3), which gives rise to a proof technique for satisfaction of modal formulae. As an immediate consequence of these properties, Corollary 6.4 shows that if states s and t in two transition systems are related by an opsimulation, every modal formula satisfied by t is also satisfied by s. We call this latter property **observational opsimilarity**.

We complete the paper in §6.1, analysing the most reasonable condition under which observational opsimilarity implies the existence of an opsimulation between the systems. The main point here is that the relation of observational opsimilarity arises from the closed structure of the bicategory of relations $\mathcal{R}el$, and we conclude that preservation of this universal property by transition relations is precisely the condition of $\mathcal{H}$-saturation of [7]. To sum up, in this paper we lay down the fundamentals of a categorical framework in which to study *simulation theory of non-deterministic systems and their logic of observable properties*.

Our general background reference for fibrations and categorical logic is [14].

Part I
A categorical analysis of relational modalities

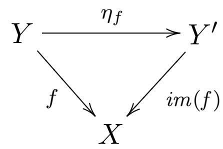
2. Bicategories of spans, relations, and partial maps: universal characterisation

We start with a brief summary of the basic categorical constructions, with details in the following subsections. Given a category $\mathbf{B}$ with pullbacks, we can construct the bicategory of spans $\mathbf{Spn}(\mathbf{B})$. If we furthermore assume a stable factorisation system $(\mathcal{E}, \mathcal{M})$ (formal epis/monos) on $\mathbf{B}$, we can construct the bicategory of relations $\mathcal{R}el(\mathbf{B})$. Recall that given a factorisation system we have for every object X in $\mathbf{B}$ a reflection

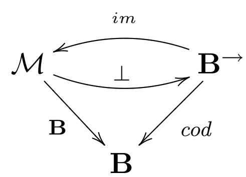
$$\mathcal{M}/X \underset{\perp}{\overset{im}{\rightleftarrows}} \mathbf{B}/X$$

where $\mathcal{M}/X$ is the full subcategory of the slice category $\mathbf{B}/X$ spanned by the $\mathcal{M}$ arrows, so that for an arrow $f : Y \to X$, we have the canonical $(\mathcal{E}, \mathcal{M})$-

factorisation



and the unit $\eta_f : Y \to Y'$ is in $\mathcal{E}$. Pullback stability of the factorisation means that the above adjunctions assemble themselves into a fibred adjunction



and we obtain $\mathcal{R}el(\mathbf{B})$ from $\mathbf{Spn}(\mathbf{B})$ by applying the reflection to the hom categories, as we make explicit below. The classical example of a stable factorisation system is given by a regular category $\mathbf{B}$ (*e.g.* a topos), considering $\mathcal{E} = \{\text{regular epimorphism}\}$ and $\mathcal{M} = \{\text{monomorphism}\}$. Yet another related bicategory can be constructed assuming a class $\mathcal{M}$ of monos in $\mathbf{B}$ closed under composition and isomorphisms and stable under pullbacks. The arrows in $\mathcal{M}$ can be then considered as the formal domains for a category of partial maps $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$. This is a locally full sub-bicategory of $\mathcal{R}el(\mathbf{B})$, consisting of those spans whose first leg is in $\mathcal{M}$. The closure properties of $\mathcal{M}$ gurantee that such spans are closed under relational composition (which agrees with span composition in this context). All three bicategories enjoy a fundamental property in common, namely that every morphism in any of them can be factorised as a right adjoint followed by a (total) function, and such adjoints satisfy the Beck-Chevalley condition (pullback stability). We now recall the details of the above constructions and characterise them universally.

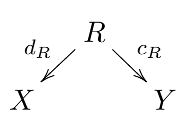
2.1. The bicategory of spans $\mathbf{Spn}(\mathbf{B})$

We start by recalling the definition of the bicategory of spans $\mathbf{Spn}(\mathbf{B})$ on a category with pullbacks $\mathbf{B}$, introduced in [2].

2.1. Definition. Given a category $\mathbf{B}$ with pullbacks, the **bicategory of spans** $\mathbf{Spn}(\mathbf{B})$ consists of

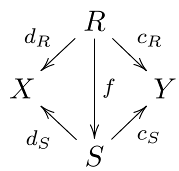
objects those of $\mathbf{B}$

morphisms a morphism from X to Y is a span

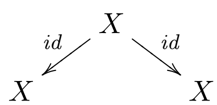


For brevity, we may write $(d_R, R, c_R) : X \to Y$ for this span.

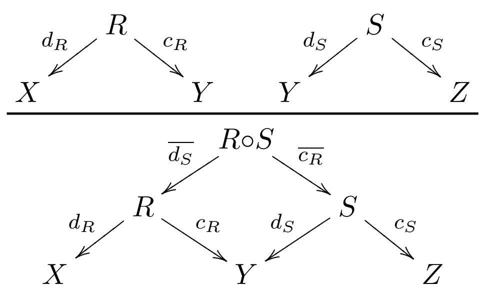
2-cells a 2-cell between morphisms is a morphism between the top objects of the spans, commuting with the domain and codomain morphisms:



The identity span on X is



and composition is given by



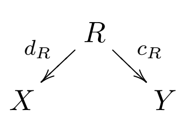
where the square is a pullback. Horizontal composition of 2-cells is clearly (canonically) induced by that of morphisms, while vertical composition is inherited from **B**.

2.2. The bicategory of relations $\mathcal{R}el(\mathbf{B})$

We now assume a stable factorisation system $(\mathcal{E}, \mathcal{M})$ on **B** [3]. The class $\mathcal{E}$ provides the abstract epimorphisms, while $\mathcal{M}$ provides the monomorphisms, generalising the classical surjective/injective function factorisation in $\mathcal{S}et$. In this context we can define the **bicategory of relations** $\mathcal{R}el(\mathbf{B})$ as follows:

objects those of **B**

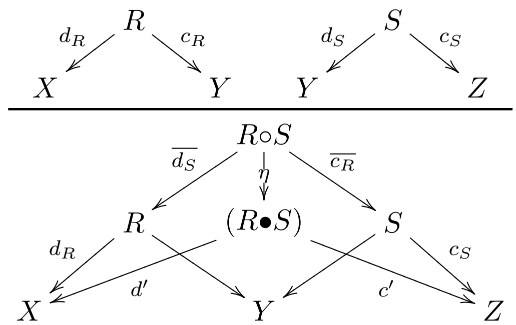
morphisms a morphism from X to Y is a span



which is a $\mathcal{M}$-arrow into $X \times Y$. We refer to such a morphism as a *relation* from X to Y, which we write $R\colon X \not\to Y$.

2-cells a 2-cell between morphisms is a morphism between the top objects of the spans, commuting with the domain and codomain morphisms as in **Spn**(**B**).

The identity relation on X is the same as in $\mathbf{Spn}(\mathbf{B})$ and composition is given by



where the square is a pullback and $\langle d', c' \rangle = im(\langle d_R \overline{d_S}, c_S \overline{c_R} \rangle) : (R \bullet S) \to X \times Z$. Horizontal composition of 2-cells is clearly (canonically) induced by that of morphisms, while vertical composition is inherited from $\mathbf{B}$. Notice that in the case of the regular epi/mono factorisation for a regular category, a relation amounts to a *jointly monic* pair of arrows, which in $\mathcal{S}et$ corresponds to a (sub)set of pairs of elements.

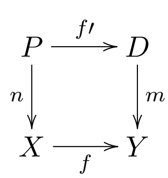
The bicategory $\mathcal{R}el(\mathbf{B})$ is related to the bicategory of spans $\mathbf{Spn}(\mathbf{B})$ as follows:

- There is an lax functor $U : \mathcal{R}el(\mathbf{B}) \to \mathbf{Spn}(\mathbf{B})$ which is the identity on objects, 1-cells and 2-cells. For composable relations $R \colon X \not\rightarrow Y$ and $S \colon X \not\rightarrow Y$, the structural 2-cell $\delta_{R,S} : U(R) \circ U(S) \Rightarrow U(R \bullet S)$ is the arrow η in the diagram defining composition in $\mathcal{R}el(\mathbf{B})$ above.
- There is an oplax functor $\rho : \mathbf{Spn}(\mathbf{B}) \to \mathcal{R}\mathbf{el}(\mathbf{B})$ which is identity on objects and sends every span (s, t) to the associated relation $(s, t)_{\#}$, while the functoriality of the factorisation determines the action on 2-cells.
- ρ is locally left adjoint to U, *i.e.* for every pair of objects X and Y, the induced functors on the hom-categories exhibit $\mathcal{R}el(\mathbf{B})(\mathbf{X}, \mathbf{Y})$ as a reflective subcategory of $\mathbf{Spn}(\mathbf{B})(\mathbf{X}, \mathbf{Y}) = \mathbf{B}/\mathbf{X} \times \mathbf{Y}$.

2.3. The bicategory of partial maps $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$

Consider a class of monos $\mathcal{M}$ in $\mathbf{B}$ satisfying:

- All isomorphisms belong to $\mathcal{M}$.
- $\mathcal{M}$ is closed under composition.
- Given a pullback square



  whenever m is in $\mathcal{M}$, so is n.

We can construct the **bicategory of partial maps** $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ as a subbicategory of $\mathbf{Spn}(\mathbf{B})$, with the same objects and 2-cells but whose morphisms are those spans

$$X \xleftarrow{\;m\;} D \xrightarrow{\;f\;} Y$$

where $m : D \to X$ is in $\mathcal{M}$. We write $(m, f)\colon X \rightharpoonup Y$ for such a morphism. The assumptions on $\mathcal{M}$ imply that such spans are closed under composition and identities. If $\mathcal{M}$ is the class of formal monos of a stable factorisation system as in §2.2, it is not difficult to verify that $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ is equally a subbicategory of $\mathcal{R}el(\mathbf{B})$.

2.4. Universal characterisations

The crucial property of $\mathbf{Spn}(\mathbf{B})$ (inherited by $\mathcal{R}el(\mathbf{B})$ and $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$) is that every span $R\colon X \not\rightarrow Y$ factors as $d_R^* \bullet (c_R)_\#$,

$$d_R^* = \; X \xleftarrow{\;d_R\;} R \xrightarrow{\;id\;} R \qquad\qquad (c_R)_\# = \; R \xleftarrow{\;id\;} R \xrightarrow{\;c_R\;} Y$$

More generally, we have two embeddings $\_{}_\# : \mathbf{B} \to \mathbf{Spn}(\mathbf{B})$ and $(\_)^* : \mathbf{B}^{op} \to \mathbf{Spn}(\mathbf{B})$ such that $f_\# \dashv f^*$ for $f : X \to Y$ in $\mathbf{B}$. These right adjoints satisfy the Beck-Chevalley condition

$$\begin{array}{ccc} P & \xrightarrow{\;q\;} & X \\ {\scriptstyle p}\downarrow & p.b. & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\;g\;]{} & Z \end{array} \qquad \Longrightarrow \qquad \Sigma_q p^* \cong f^* \Sigma_g$$

In the context of bicategories, it is standard to call a 1-cell with a right adjoint a **map** (a terminology introduced by Lawvere).

In the case of $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$, only the morphisms in $\mathcal{M}$ are sent by $\_{}_\#$ to maps. A further property, distinctive of $\mathcal{R}el(\mathbf{B})$, is that for $e : X \to Y$ in $\mathcal{E}$,

$$e^* \bullet e_\# = id\colon Y \not\rightarrow Y.$$

Consider the (pseudo-)functor $\eta : \mathbf{B} \to \mathcal{X}$ (where $\mathcal{X}$ stands for any of $\mathbf{Spn}(\mathbf{B})$, $\mathcal{R}el(\mathbf{B})$ or $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$) given by:

$$X \xrightarrow{\;f\;} Y \qquad \longmapsto \qquad X \xleftarrow{\;id\;} X \xrightarrow{\;f\;} Y$$

2.2. Theorem (Universal characterisation of $\mathbf{Spn}(\mathbf{B})$[11, Thm. A.2]). *Consider a category $\mathbf{B}$ with pullbacks*

1. *The (pseudo-)functor $\eta_{\mathbf{B}} : \mathbf{B} \to \mathbf{Spn}(\mathbf{B})$ is universal among pseudo-functors from $\mathbf{B}$ to bicategories $\mathcal{K}$, $F : \mathbf{B} \to \mathcal{K}$, which send the morphisms of $\mathbf{B}$ to **maps**, satisfying the Beck-Chevalley condition.*

2. *Given two such pseudo-functors $F, G : \mathbf{B} \to \mathcal{K}$ and a pseudo-natural transformation $\alpha : F \Rightarrow G$, there is a unique lax transformation $\hat{\alpha} : \hat{F} \Rightarrow \hat{G}$ such that $\hat{\alpha}\eta_{\mathbf{B}} = \alpha$. Furthermore, if for every morphism $f : X \to Y$ in $\mathbf{B}$ the pair (α_X, α_Y) induces a pseudo-map of adjoints from $Ff \dashv (Ff)^*$ to $Gf \dashv (Gf)^*$, the corresponding 2-cell $\hat{\alpha}$ is pseudo-natural as well.*

Universality in (1) means that precomposition with $\eta_{\mathbf{B}}$ induces an equivalence of bicategories

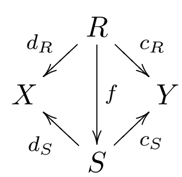
$$\_ \circ \eta_{\mathbf{B}} : \mathsf{Hom}(\mathbf{Spn}(\mathbf{B}), \mathcal{K}) \to \mathsf{Hom}_{mapBC}(\mathbf{B}, \mathcal{K})$$

pseudo-natural in $\mathcal{K}$, where $\mathsf{Hom}(\mathbf{Spn}(\mathbf{B}), \mathcal{K})$ denotes the bicategory of pseudo-functors, pseudo-natural transformations and modifications (as in [21]) and $\mathsf{Hom}_{mapBC}(\mathbf{B}, \mathcal{K})$ is the corresponding sub-bicategory of pseudo-functors sending morphisms of $\mathbf{B}$ to maps satisfying the Beck-Chevalley condition, and psuedo-natural transformations inducing pseudo-maps of adjoints, as in (2) above.

Proof. The homomorphism $\hat{F} : \mathbf{Spn}(\mathbf{B}) \to \mathcal{K}$ preserves adjoints, hence the factorisation of spans above implies that $\hat{F}$ is determined as

$$\hat{F}(R\colon X \not\to Y) = \hat{F}(d_R^* \bullet (c_R)_\#) = (Fd_R)^* \bullet (Fc_R)$$

The Beck-Chevalley condition ensures that $\hat{F}$ so defined does preserve composition in $\mathbf{Spn}(\mathbf{B})$ up to coherent isomorphism. As for the action of $\hat{F}$ on 2-cells, given



let

$$\hat{F}f = (Fd_S)^* \epsilon_{Fh}(Fc_S) : (Fd_R)^* \bullet (Fc_R) \Rightarrow (Fd_S)^* \bullet id \bullet (Fc_S)$$

where $\epsilon_{Fh} : (Fh)^* \bullet Fh \Rightarrow id$ is the counit of the adjunction for the map Fh. Notice that we have left implicit the use of the (structural) isomorphisms $(Fd_R)^* \cong (Fd_S)^* \bullet (Ff)^*$ and $Fc_R \cong Ff \bullet Fc_S$. The extension of this assignment to 2-cells between pseudo-functors (and modifications) is a routine pasting calculation. □

With the same bicategorical universality criteria as stated in the theorem above, we have the following variants characterising bicategories of relations and partial maps.

2.3. Theorem (Universal characterisation of $\mathcal{R}el(\mathbf{B})$). *Consider a category $\mathbf{B}$ with pullbacks and a stable factorisation system $(\mathcal{E}, \mathcal{M})$. The pseudo-functor $\eta_{\mathbf{B}} : \mathbf{B} \to \mathcal{R}\mathbf{el}(\mathbf{B})$ is universal among pseudo-functors from $\mathbf{B}$ to bicategories $\mathcal{K}$, $F : \mathbf{B} \to \mathcal{K}$, which send the morphisms of $\mathbf{B}$ to* ***maps*** *(1-cells with a right adjoint) satisfying the Beck-Chevalley condition, and such that the right adjoint e^* is additionally a left inverse when the morphism e is in $\mathcal{E}$.*

Proof. The homomorphism $\hat{F} : \mathcal{R}el(\mathbf{B}) \to \mathcal{K}$ is determined exactly as in the proof of Theorem 2.2. The only point to notice is that the additional condition about right adjoints for Fe when e is in $\mathcal{E}$ guarantees that $\hat{F}$ is indeed pseudo-functorial for composition in $\mathcal{R}el(\mathbf{B})$. □

2.4. Theorem (Universal characterisation of $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$). *Consider a category* $\mathbf{B}$ *with pullbacks and a class* $\mathcal{M}$ *of monos as in §2.3.*

1. *The pseudo-functor* $\eta_{\mathbf{B}} : \mathbf{B} \to \mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ *is universal among pseudo-functors from* $\mathbf{B}$ *to bicategories* $\mathcal{K}$, $F : \mathbf{B} \to \mathcal{K}$, *which send the morphisms of* $\mathcal{M}$ *to* ***maps*** *(1-cells with a right adjoint) satisfying the Beck-Chevalley condition.*
2. *Given two such pseudo-functors* $F, G : \mathbf{B} \to \mathcal{K}$ *and a pseudo-natural transformation* $\alpha : F \Rightarrow G$, *there is a unique lax transformation* $\hat{\alpha} : \hat{F} \Rightarrow \hat{G}$ *such that* $\hat{\alpha}\eta_{\mathbf{B}} = \alpha$. *Furthermore, if for every morphism* $m : X \to Y$ *in* $\mathcal{M}$ *the pair* (α_x, α_y) *induces a pseudo-map of adjoints from* $Fm \dashv (Fm)^*$ *to* $Gm \dashv (Gm)^*$, *the corresponding 2-cell* $\hat{\alpha}$ *is pseudo-natural as well.*

2.5. Remark. In [4] the authors stablish an appropriate variant of the universal property in Theorem 2.2 and examine further consequences. The 1-dimensional version of the universal property of $\mathbf{Spn}(\mathbf{B})$ as a bicategory seems to be folklore, as it mentioned in passing (without any precise details) in [17], which provides an explicit description of the free addition of pullbacks to a category. The variant characterizing $\mathcal{R}el(\mathbf{B})$ seems original with us, while that for $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ is presumably known, although we know of no references. The 2-dimensional aspect of the above universal properties seems to be stated here for the first time.

3. Categorical logic view of relational modalities

Consider a category $\mathbf{B}$ with pullbacks and a fibration $p : \mathbf{P} \to \mathbf{B}$ admitting sums, *i.e.* left adjoints to substitution functors satisfying the Beck-Chevalley condition. Logically, we think of the objects of $\mathbf{B}$ as *types*, and its morphisms as (equivalence classes of) *terms*. The objects of $\mathbf{P_X}$ (the fibre over X) correspond to *predicates* (on the type X), $x : X \mid \phi$, while morphisms correspond to equivalence classes of *entailments* (or constructive proofs) between such predicates $x : X \mid \phi \vdash \psi$. For a morphism/term $t : X \to Y$, the functor $t^* : \mathbf{P_Y} \to \mathbf{P_X}$ is interpreted as sending a predicate $y : Y \mid \phi$ to $x : X \mid \phi(tx/y)$, *i.e.* performing substitution of the term t for the variable y. The *sum* or *direct image* $\Sigma_t : \mathbf{P_X} \to \mathbf{P_Y}$ corresponds to a 'generalised existential quantification: it can be expressed in terms of ordinary existential quantification, equality and conjunction as follows:

$$x : X \mid \phi \xmapsto{\Sigma_t} y : Y \mid \exists x : X.\ tx = y \wedge \phi$$

We also consider the dual situation where the fibration $p : \mathbf{P} \to \mathbf{B}$ admits products, that is, the existence of right adjoints to substitution satisfying Beck-Chevalley. Logically, such a product functor $\Pi_t : \mathbf{P_X} \to \mathbf{P_Y}$ corresponds to a generalised universal quantification, which is expressible in terms of ordinary universal quantifiers, equality and implication:

$$x\,:\,X \mid \phi \xmapsto{\Pi_t} y\,:\,Y \mid \forall x\,:\,X.\; tx = y \Longrightarrow \phi.$$

A fibration with sums gives rise to a contravariant pseudo-functor $F^p : \mathbf{B}^{op} \to \mathcal{C}at$, as well as a covariant one $F_p : \mathbf{B} \to \mathcal{C}at$ using the sums $\Sigma_{(\_)}$. Furthermore, when $\mathbf{B}$ is endowed with a stable factorisation system $(\mathcal{E}, \mathcal{M})$ as in §2, we require[1] $\Sigma_e e^* \cong id$ whenever $e \in \mathcal{E}$. Logically, this means that $e : X \to Y$ is *surjective* (has entire image) as far as the logic embodied by $p : \mathbf{P} \to \mathbf{B}$ is concerned:

$$y\,:\,Y \vdash (\exists x\,:\,X.\; ex = y \wedge \phi(ex/y)) \equiv \phi$$

As for the canonical example of the regular epi/mono factorisation in a regular category $\mathbf{B}$, the *internal logic* fibration $cod : Sub(\mathbf{B}) \to \mathbf{B}$ (the objects of the total category being subobjects) satisfies all these properties. For a fibration $p : \mathbf{P} \to \mathbf{B}$ with the above properties, the associated covariant functor $F_p : \mathbf{B} \to \mathcal{C}at$ satisfies the hypothesis of Theorem 2.3 and thus induces a homomorphism

$$\hat{F}_p : \mathcal{R}el(\mathbf{B}) \to \mathcal{C}at$$

Since $\mathcal{R}el(\mathbf{B})$ is self-dual (simply by turning the relations around), we also get a contravariant homomorphism, denoted the same way, $\hat{F}_p : \mathcal{R}el(\mathbf{B})^{op} \to \mathcal{C}at$, with action

$$(X \xleftarrow{d_R} R \xrightarrow{c_R} Y) \qquad \mapsto \qquad \mathbf{P_Y} \xrightarrow[c_R^*]{} \mathbf{P_R} \xrightarrow[\Sigma_{d_R}]{} \mathbf{P_X}$$

which, in the logical interpretation of these operations, reads as

$$y\,:\,Y \vdash \phi \longmapsto x\,:\,X \vdash \exists r\,:\,R.\; d_R r = x \wedge \phi(c_R r/y)$$

In more detail, consider the case $\mathcal{R}el = \mathcal{R}el(\mathcal{S}et)$ the usual category of sets and relations. A relation $R\colon X \not\to Y$ is a set of pairs (x, y) and

$$\exists r\,:\,R.\; d_R r = x \wedge \phi(c_R r/y) \quad \equiv \quad \exists x\,:\,X.\; xRy \wedge \phi(y)$$

and the above formula for relational substitution becomes

$$y\,:\,Y \vdash \phi(y) \longmapsto x\,:\,X \vdash \exists y\,:\,Y. xRy \wedge \phi(y)$$

which is none other than the formula interpreting $\langle R\rangle\phi$ at x in (first-order) predicate logic *cf.*[7, 6]. The other standard relational modality $\Box$ is obtained by duality, considering the (fibrewise) dual fibration $p^{vop} : \mathbf{P}^{vop} \to \mathbf{B}$. Let us assume that the original fibration admits products such that for $e \in \mathcal{E}$, $\Pi_e e^* \cong id$ which as far as the logic embodied by p goes, means that the morphism e has

[1] This requirement admits an intrinsic formulation, without reference to a choice of substitution functors: a cartesian morphism over $e \in \mathcal{E}$ must also be cocartesian.

non-empty fibers; this is yet another way of demanding surjectivity. Under these assumptions, the pseudofunctor associated to p^{vop} extends to the bicategory $\mathcal{R}el(\mathbf{B})$ and we can apply the above interpretation to the resulting substitution functor:

$$y : Y \mid \phi(y) \longmapsto x : X \mid \forall r : R.\ d_R r = x \Rightarrow \phi(c_R r/y)$$

which in $\mathcal{R}el$ is equivalent to

$$y : Y \mid \phi(y) \longmapsto x : X \mid \forall y : Y.\ xRy \Longrightarrow \phi(y)$$

which is the (first-order) logic interpretation of $[R]\phi$ at x. It is interesting to point out the special case when the relation $R\colon X \not\rightarrow Y$ is a partial map $f\colon X \rightharpoonup Y$ (specified by a 'subobject' $dom(f)$ of X and a morphism $f : dom(f) \to Y$. The possibility modality $\langle f\rangle$ amounts to the following:

$$y : Y \mid \phi \longmapsto x : X \mid x \in dom(f) \wedge \phi(fx/y)$$

which is the interpretation of substitution of terms in predicates in Fourman-Scott *logic of partial terms* [5, 20]. The necessity modality $[f]$ has the following interpretation:

$$y : Y \mid \phi(y) \longmapsto x : X \mid x \in dom(f) \Longrightarrow \phi(fx/y)$$

which is *Dijkstra's weakest precondition* operator for *Hoare triples*, if we regard the partial map f as the denotation of a program transforming states *cf.*[6, Ch.7].

3.1. Remark (*Exactness properties of modal operators*). The decomposition of $\langle \_ \rangle$ as $\Sigma_{d_R} \circ c_R^*$ means that it preserves those colimits stable under substitution (since Σ_{d_R} is a left adjoint, it preserves whichever colimits exist). In particular $\langle \_ \rangle$ preserves joins. Likewise, other exactness properties can be inferred from this decomposition, *e.g.* $\langle R\rangle\top \equiv \top$ iff d_R is epi.

3.2. Remark (*Monadic interpretation of* $\langle \_ \rangle$). In [1] the $\langle \_ \rangle$-modality is modeled as a monad with some additional structure, on a so-called *CS4 category*, generalizing closure operators on a Heyting algebra. This structure is accounted for by our analysis as follows: a preorder is a reflexive and transitive relation. Categorically, this structure corresponds to a **monad** in the bicategory $\mathcal{R}el$. This amounts in turn to a lax functor $M : \mathbf{1} \to \mathcal{R}el$, and composing it with the pseudo-functor $\hat{F}_p : \mathcal{R}el^{op} \to \mathcal{C}at$ we obtain a monad (in $\mathcal{C}at$) on the fibre over the domain (= codomain) of R. In fact, in this situation, since $\hat{F}_p$ is full and faithful on 2-cells, for a relation $R\colon X \not\rightarrow X$, the associated functor $\langle R\rangle : \wp(X) \to \wp(X)$ bears a monad structure if and only if R is a preorder. Notice however that the relations of a transition system (see §4) are seldom preorders, hence the associated modalities cannot be modeled by (co)monads, and we may conclude that our analysis obtains a more fundamental set of categorical primitives to study these phenomena.

3.3. Remark (Forward vs. Backward modalities). Given that $\langle R\rangle = \Sigma_{d_R} \circ {c_R}^*$ and $[R] = \Pi_{d_R} \circ {c_R}^*$, considering the dual relation R^o (switching domain and codomain) we have the following adjunctions:

$$\langle R\rangle \dashv [R^o] \qquad \langle R^o\rangle \dashv [R]$$

The modalities $\langle R^o\rangle$ and $[R^o]$ are usually referred to as *backward*, since they consider the relation R with its opposite orientation. In [19], the authors introduce a modal logic, with intended applications in epistemic systems, which puts these *adjoint modalities* at the forefront.

3.1. Fibrations of spans, relations and partial maps

Having analysed the extension of first-order logic from *functions* to *relations* as an extension of pseudo-functors, we spell out the resulting fibration obtained via the usual Grothendieck construction. This is a mere auxiliary step in order to analyse representability in §3.2. The reader uncomfortable with the interplay between fibrations and pseudo-functors could simply look up the resulting *logical predicate* formulae for powerobjects and lifting (partial map classifier) that result from this process. Starting with a fibration $p : \mathbf{P} \to \mathbf{B}$ with sums, such that $\Sigma_e e^* \cong id$ for morphisms e in $\mathcal{E}$, we have seen in §3 how to extend the associated pseudofunctor $F_p : \mathbf{B}^{op} \to \mathcal{C}at$ to one $\hat{F}_p : \mathbf{Spn}(\mathbf{B})^{op} \to \mathcal{C}at$ according to the various conditions required on F_p. Applying the Grothendieck construction to the pseudo-functor $\hat{F}_p : \mathbf{Spn}(\mathbf{B})^{op} \to \mathcal{C}at$ we obtain a bicategory $\mathcal{S}\mathbf{P}$, a homomorphism $\mathcal{S}p : \mathcal{S}\mathbf{P} \to \mathbf{Spn}(\mathbf{B})$ and a morphism $\eta_p : p \to \mathcal{S}p$ of fibrations (over $\eta_{\mathbf{B}} : \mathbf{B} \to \mathbf{Spn}(\mathbf{B})$). In more detail, $\mathcal{S}\mathbf{P}$ consists of

objects those of $\mathbf{P}$

morphisms a morphism from P (in $\mathbf{P_X}$) to Q (in $\mathbf{P_Y}$) is a pair $(f, (d, R, c))$, where $(d, R, c) : X \to Y$ is a 1-cell in $\mathbf{Spn}(\mathbf{B})$ and $f : P \to \Sigma_d c^* Q$ in $\mathbf{P_X}$. We can also give an intrinsic description of the resulting morphisms (without recourse to an explicit choice of substitution functors) as follows: a morphism in $\mathcal{S}\mathbf{P}(\mathbf{P}, \mathbf{Q})$ is an equivalence class of pairs $\langle f, (\underline{d}, T, \overline{c})\rangle$ as displayed

$$P \xrightarrow{f} S \xleftarrow{\underline{d}} T \xrightarrow{\overline{c}} Q \tag{1}$$

where $pf = id_{pP}$, $\overline{c}$ is cartesian and $\underline{d}$ is cocartesian. The equivalence relation between such triples involves vertical isomorphims which make the morphism completely determined by the underlying span and the equivalence class of f. That is, different choices of cartesian map $\overline{c}' : T' \to Q$ and cocartesian map $\underline{d}' : T' \to S'$ determine canonical vertical isomorphisms $t : T \xrightarrow{\sim} T'$ and $s : S \xrightarrow{\sim} S'$ which in turn determine the equivalence of the pairs

$$\langle f, (\underline{d}, T, \overline{c})\rangle \simeq \langle tf, (\underline{d}', T', \overline{c}')\rangle$$

2-cells a 2-cell between morphisms $(f, (d, R, c))$ and $(g, (d', R', c'))$ is simply a 2-cell between the spans (d, R, c) and (d', R', c') in $\mathbf{Spn}(\mathbf{B})$.

The evident homomorphism $\mathcal{S}p : \mathcal{S}\mathbf{P} \to \mathbf{Spn}(\mathbf{B})$ induces a functor between the associated *classifying categories* (in the sense of [2]), which is clearly a fibration. The functor $\tilde{\eta}_{\mathbf{P}} : \mathbf{P} \to \mathcal{S}\mathbf{P}$ acts as the identity on objects, while its action on morphisms is as follows: given a morphism, consider a (vertical, cartesian) factorisation and map it to the pair consisting of the vertical part and the associated representable span of the cartesian morphism. Diagrammatically,

$$
\begin{array}{ccc}
P \xrightarrow{\;f\;} Q
& \mapsto &
\begin{array}{c} P \\ {\scriptstyle \hat{f}}\downarrow \\ \overline{Q} \xrightarrow{\;\overline{pf}\;} Q \end{array}
& \mapsto &
\begin{array}{c} P \qquad\qquad \overline{Q} \\ {\scriptstyle \hat{f}}\downarrow \;\swarrow{\scriptstyle id} \qquad \searrow{\scriptstyle \bar{c}} \\ \overline{Q} \qquad\qquad\qquad Q \end{array}
\end{array}
$$

Likewise, when the fibration $p : \mathbf{P} \to \mathbf{B}$ satisfies the relevant additional conditions (regarding the formal epis), the extensions $\hat{F}_p : \mathcal{R}el(\mathbf{B})^{op} \to \mathcal{C}at$ and $\hat{F}_p : \mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})^{op} \to \mathcal{C}at$ give rise to fibrations $\mathcal{R}p : \mathcal{R}\mathbf{P} \to \mathcal{R}el(\mathbf{B})$ and $\mathsf{Ptl}_{\underline{\mathcal{M}}}p : \mathsf{Ptl}_{\underline{\mathcal{M}}}\mathbf{P} \to \mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ with entirely analogous descriptions to the one above (1).

3.4. Remark. Notice that we have considered above the $\langle \_ \rangle$ modality as the action of a span/relation/partial map, as it lends itself to a direct intrinsic description of the resulting spans via cocartesian maps. By applying the same construction to the (fibrewise) dual fibration $p^{vop} : \mathbf{P}^{vop} \to \mathbf{B}$, the action is then that of the $\Box$ modality.

3.2. Representability

3.2.1. Relations and their powerobject

Having analysed in §2 the constructions $\mathcal{R}el(\mathbf{B})$ and $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$, we consider now the situation when such concepts, *viz.* relations and partial maps, are *representable* in $\mathbf{B}$. Representability of relations amounts to the requirement that the functor $\eta_{\mathbf{B}} : \mathbf{B} \to \mathcal{R}el(\mathbf{B})$ admit a right adjoint $\wp : \mathcal{R}el(\mathbf{B}) \to \mathbf{B}$, which is none other than the *power-object* functor, characteristic of elementary toposes [15]. The counit of the adjunction is the *membership* relation $\varepsilon_X : \wp X \not\rightarrow X$. In this situation we have the following diagram:

$$
\begin{array}{ccc}
\mathbf{P} & \xrightarrow{\;\tilde{\eta}_p\;} & \mathcal{R}\mathbf{P} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathcal{R}p} \\
\mathbf{B} & \underset{\wp}{\overset{\eta_{\mathbf{B}}}{\rightleftarrows}}\;{\scriptstyle \perp} & \mathcal{R}el(\mathbf{B})
\end{array}
$$

and the square is a pullback. That is to say, we recover the original fibration p considering the relational modalities of *functional relations*.

Now we have the set-up of [10, Lemma 4.1], and we conclude that $\tilde{\eta}_p$ has a right adjoint $\underline{\wp} : \mathcal{R}\mathbf{P} \to \mathbf{P}$, obtained by reindexing against the counit of the adjunction at the base, $\underline{\wp}(P) \cong \varepsilon^*(P)$. In logical terms, the action of $\underline{\wp}(P)$ is

$$y \,:\, Y \mid \phi(y) \longmapsto S \,:\, \wp(Y) \mid \exists y \in Y.\, y \varepsilon S \wedge \phi(y)$$

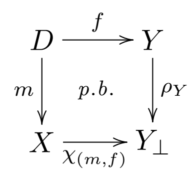
If we consider the same construction applied to the dual fibration p^{vop} we obtain

$$y : Y \mid \phi(y) \longmapsto S : \wp(Y) \mid \forall y \in Y.\, y \varepsilon S \Longrightarrow \phi(y)$$

We thus obtain in each case a monad $\tilde{\wp} : \mathbf{P} \to \mathbf{P}$ fibred over $\wp : \mathbf{B} \to \mathbf{B}$, which yields a categorical version of a **logical predicate for the powerobject** in the framework of [8].

3.2.2. Partial-map classifier

We carry out a similar analysis for the partial map classifier, *viz.* the right adjoint to $\eta_{\mathbf{B}} : \mathbf{B} \to \mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$. Partial map classifiers exist in elementary toposes [15] and in most standard categories of domains. Recall that a partial map classifier for maps with codomain Y is an object $Y_\bot$ equipped with a *generic domain of partiality* $\rho_Y : Y \hookrightarrow Y_\bot$ such that for any partial map $(m, f)\colon X \rightharpoonup Y$, there is a unique morphism $\chi_{(m,f)} : X \to Y_\bot$ in $\mathbf{B}$ through which we obtain (m, f) as a pullback:



The counit of the adjunction is thus the partial map $\rho^*_Y : Y_\bot \to Y$: $\eta^*_Y(\rho_Y(y)) = y$ and undefined for those elements in $Y_\bot$ not in the image of ρ_Y (this is unambiguous since ρ_Y is monic). Since the morphism of fibrations $(\tilde{\eta}_p, \eta_{\mathbf{B}}) : p \to \mathsf{Ptl}_{\underline{\mathcal{M}}}p$ is cartesian, a right adjoint $\underline{(\_)_\bot} : \mathsf{Ptl}_{\underline{\mathcal{M}}}(\mathbf{P}) \to \mathbf{P}$ to $\tilde{\eta}_{\mathbf{P}} : \mathbf{P} \to \mathsf{Ptl}_{\underline{\mathcal{M}}}(\mathbf{P})$ is given by reindexing against the counit of the base adjunction ([11, Lemma 4.1]): $\underline{(\_)_\bot}(P) \cong \langle \rho^*_{pP} \rangle (P)$, which logically amounts to the following formula:

$$y : Y \mid \phi(y) \longmapsto \bar{y} : Y_\bot \mid \bar{y} \downarrow \wedge \phi(y)$$

where we have used the convention that y is the (unique) value in Y of $\bar{y} \in Y_\bot$ when $\bar{y}$ is in the image of $\rho_Y : Y \to Y_\bot$ (which is the meaning of the predicate $\_ \downarrow$).

If we consider the same situation for the dual fibration p^{vop}, we obtain the following formula for $\underline{(\_)_\bot} : \mathsf{Ptl}_{\underline{\mathcal{M}}}(\mathbf{P}) \to \mathbf{P}$:

$$y : Y \mid \phi(y) \longmapsto \bar{y} : Y_\bot \mid \bar{y} \downarrow \Longrightarrow \phi(y)$$

We have thus deduced two possible versions of **logical predicate for the lifting monad** according to our framework [8]. The second action has a neat intrinsic interpretation, in terms on the internal logic of the (bi)category of partial maps, which we explain in the following subsection.

3.3. Internal logic of partial maps

A staple of categorical logic is the interpretation of logical systems internally in a category $\mathbf{B}$ as a *universe of discourse*. Undoubtedly, the biggest achievement in this direction has been the development of topos theory; a substantial body of mathematics has been (re)developed inside an elementary topos $\mathcal{E}$ as an abstract substitute for the category of sets, the traditional universe of discourse for mathematics. For instance, both relations and partial maps are representable in a topos [15].

In this context, a *predicate* ϕ is interpreted as a *subobject* $m_\phi : D \hookrightarrow X$, a *term* $\Gamma \vdash t : \sigma$ is interpreted as a morphism $t : G \to S$ (where G interprets the product of the types in Γ and S interprets the type σ) and the substitution $\phi(t/y)$ corresponds to the pullback of m_ϕ along t, $t^*(m_\phi) : t^*D \hookrightarrow G$. Thus, the internal logic of $\mathbf{B}$ amounts essentially to the structure present in the subobject fibration $cod : \mathsf{Sub}(\mathbf{B}) \to \mathbf{B}$, where pullbacks provide cartesian liftings (substitutions).

Let us examine what this amounts to for a category of partial maps $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$. Firstly, let us notice that subobjects in this category are exactly those of $\mathbf{B}$:

3.5. Lemma. *A partial map $(m, f)\colon X \rightharpoonup Y$ is monic iff m is an isomorphism and f is monic in* $\mathbf{B}$

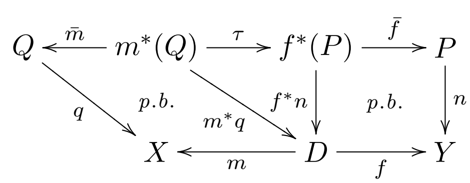
Proof. If $(m, f) = (id, f) \circ (m, id)$ is monic, so is (m, id). But $(m, id) \circ (id, m) = (id, id)$, so (m, id) is always a split epimorphism, hence it must be an isomorphism, with (id, m) as inverse. So $(m, m) = (id, m) \circ (m, id) = (id, id)$ and m is an isomorphism.

Now, $(m, f) = (id, f \circ m^{-1})$ and postcomposition with the latter is just like in $\mathbf{B}$, so f must be a monomorphism. The converse is immediate. □

Secondly, let us examine what pullbacks of such monos amount to:

3.6. Proposition. $cod : \mathsf{Sub}(\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})) \to \mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$ *is a fibration: the pullback of a mono $n : P \hookrightarrow Y$ along a partial map $(m, f)\colon X \rightharpoonup Y$ is* $\Pi_m(f^*n)$

Proof. Let us spell out the putative cartesian morphism:



where $q = \Pi_m(f^*n)$ and $\tau : m^*(Q) \to f^*(P)$ is the counit of the adjunction $m^* \dashv \Pi_m$ at $f^*(n)$. Given a mono $r : R \hookrightarrow X$ and a partial map $(s, t)\colon R \rightharpoonup P$ (with source S) such that $(m, f) \circ r = m \circ (s, t)$ or equivalently $\theta : S \xrightarrow{\sim} m^*(R)$, we get a unique mediating morphism $l : S \hookrightarrow f^*n$ such that $\bar{f} \circ l = t$ and $f^*n \circ l = m^*r \circ \theta$. Finally, transposing the composite $l \circ \theta^{-1} : m^*r \to f^*n$ across the adjunction $m^* \dashv \Pi_m$, we obtain the required unique mediating (mono)morphism $h : r \to q$. □

We conclude then that, since $[(m, f)]$ substitution action provides the relevant pullbacks for the internal logic of subobjects, it should be considered as more fundamental than its classical dual $\langle (m, f) \rangle$.

3.7. Remark. It is easy to verify, with the data in the proof of Proposition 3.6, that if either

- $\mathbf{B}$ admits image factorizations, or
- $m^* \dashv \Pi_m : \mathbf{B}/\mathbf{D} \to \mathbf{B}/\mathbf{X}$ (*e.g.* when $\mathbf{B}$ is locally cartesian closed)

then the diagram is actually a pullback in $\mathsf{Ptl}_{\mathcal{M}}(\mathbf{B})$

Part II

Satisfaction of modal formulae vs. Opsimulation

4. Modal formulae over a transition system

In this section we recall the basic background about transition systems, simulations and modal formulae. A **transition system** $\underline{S}$ over a set of labels (or 'actions') L consists of:

- a set S, whose elements are referred to as the *states* of the system
- an L-indexed family of relations $\{\alpha_{\underline{S}}: S \not\rightarrow S\}_{\alpha \in L}$ on S ($\alpha_{\underline{S}} \subseteq S \times S$)

We write $\underline{S} = (S, \{\alpha_{\underline{S}}\}_{\alpha \in L})$ for such data; $s \xrightarrow{\alpha} s'$ stands for $s(\alpha_{\underline{S}})s'$, that is, the states s, s' are related by $\alpha_{\underline{S}}$. Given two transition systems $\underline{S} = (S, \{\alpha_{\underline{S}}\}_{\alpha \in L})$ and $\underline{T} = (T, \{\alpha_{\underline{T}}\}_{\alpha \in L})$, a **simulation** between them is given by a relation $\rho: T \not\rightarrow S$ such that

$$\forall \alpha \in L.\, t(\rho)s \wedge t \xrightarrow{\alpha} t' \Longrightarrow \exists s' \in S.\, s \xrightarrow{\alpha} s' \wedge t'(\rho)s'.$$

Following [18, 7] we consider modal formulae to examine labelled transition systems. The major difference in our approach is that we consider the *intuitionistic version* of this setting. That is, we consider that our observable properties form a **frame** as Abramsky-Vickers [22]. Modal formulae are given (generated) by the syntax

$$\phi ::= \top \mid \bot \mid \phi \wedge \phi' \mid \bigvee_{i \in I} \phi_i \mid \langle \alpha \rangle \phi$$

thus equivalence classes of formulae form a **f**rame Φ (finite conjunctions which distribute over arbitrary disjunctions) and every label α has associated a *possibility* modality $\langle \alpha \rangle : \Phi \to \Phi$, which preserves all sups. We interpret modal formulae in transition systems, that is, we take transition systems as **models** of modal logic. The **s**atisfaction relation $\models_{\underline{S}}: S \not\rightarrow \Phi$ is defined by the following clauses:

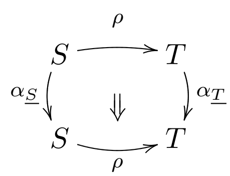
$$\begin{array}{lllll} s & \models & \top & & \\ s & \not\models & \bot & & \\ s & \models & \phi \wedge \phi' & \text{iff} & s \models \phi \quad \text{and} \quad s \models \phi' \\ s & \models & \bigvee_{i \in I} \phi_i & \text{iff} & s \models \phi_i \quad \text{, for some } \; i \in I \\ s & \models & \langle \alpha \rangle \phi & \text{iff} & s' \models \phi \quad \text{, for some } \; s' \; \text{ such that } \; s \xrightarrow{\alpha} s' \end{array}$$

5. Opsimulations as lax-transformations

We consider now the dual notion to that of simulation, which we call **opsimulation**: given transition systems $\underline{S} = (S, \{\alpha_{\underline{S}}\}_{\alpha \in L})$ and $\underline{T} = (T, \{\alpha_{\underline{T}}\}_{\alpha \in L})$, an opsimulation from $\underline{S}$ to $\underline{T}$ is given by a relation $\rho \colon S \not\rightarrow T$ such that

$$\forall \alpha \in L.\, s(\rho)t \wedge t \xrightarrow{\alpha} t' \Longrightarrow \exists s' \in S.\, s \xrightarrow{\alpha} s' \wedge s'(\rho)t'.$$

This latter expression admits a neat diagrammatic interpretation in the bicategory $\mathcal{R}el$ of sets and relations:



for every label $\alpha \in L$ The family of relations $\{\alpha_{\underline{S}}\}_{\alpha \in L}$ can be considered as a function $(\_)_{\underline{S}} : L \to \mathcal{R}el(S, S)$ and thus as a functor $\sigma_{\underline{S}} : L^* \to \mathcal{R}el$, where L^* is the free monoid on the set L construed as a one-object category. The above diagram means that:

An opsimulation $\rho \colon S \not\rightarrow T$ amounts to a lax-transformation $\rho : \sigma_{\underline{S}} \Rightarrow \sigma_{\underline{T}} : L^* \to \mathcal{R}el$.

An immediate consequence is that opsimulations compose qua lax transformations (pasting of lax-squares), hence they are closed under relational composition.

6. Modal formulae as a transition system and satisfaction as opsimulation

Consider a transition system $\underline{S}$ with set of labels L, construed as a functor $\sigma_S : L^* \to \mathcal{R}el$. The extension of the ordinary logic of sets (the fibration of subobjects $\iota : Sub(\mathcal{S}et) \to \mathcal{S}et$ to relations yields a functor[2] $F_\iota : \mathcal{R}el \to \mathsf{Frm}$ (by Theorem 2.3), where Frm denotes the (2-)category of frames and sup-preserving morphisms between them, locally ordered. The composite $F_\iota \circ \sigma_S : L^* \to \mathsf{Frm}$ yields a new transition system, whose set of states is a frame (of subsets or predicates). We denote this transition system $\underline{\mathcal{P}S} = (\mathcal{P}S, \{\langle\alpha\rangle^*\}_{\alpha \in L})$ on the frame of subsets of S (identifying predicates on a set with their extents), with action

$$\langle\alpha\rangle^* = \{(\phi, \psi) \mid \phi \equiv \langle\alpha\rangle\psi\}$$

which restricts to a transition system on the set of modal formulae Φ. In §4 we defined the satisfaction relation between S and Φ. We have just endowed Φ with the structure of a transition system and we can now characterise the satisfaction relation as an opsimulation:

[2]Since the fibres are posets, the coherent isomorphisms of the pseudo-functor are identities.

6.1. Proposition. *For a transition system $\underline{S} = (S, \{\alpha_S\}_{\alpha \in L})$, the satisfaction relation $\models_S : S \not\rightarrow \Phi$ satisfies: $\forall \alpha \in L$*

$$\begin{array}{ccc} S & \xrightarrow{\models} & \Phi \\ {\scriptstyle \alpha_{\underline{S}}} \downarrow & \cong & \downarrow {\scriptstyle \langle\alpha\rangle^*} \\ S & \xrightarrow[\models]{} & \Phi \end{array}$$

$$s \models \langle\alpha\rangle \quad \textit{iff} \quad \exists s'.s \xrightarrow{\alpha} s' \& s' \models \phi$$

Thus,$\models_S : S \not\rightarrow \Phi$ *is an opsimulation.*

To complete our purported characterisation of satisfaction, we need the following auxiliary notions:

6.2. Definition.

- Given a set S and a sup-lattice F, a relation $\rho: S \not\rightarrow F$ is called a σ-**relation** if its transpose $\hat{\rho} : F \to \mathcal{P}S$ is a sup-lattice homomorphism (where $\hat{\rho}(f) = \{s \mid (s, f) \in \rho\}$).
- Given transition systems (S, α) and (F, β) (over the same set of labels L), with F a sup-lattice, a relation $\rho \subseteq S \times F$ is called a σ-**opsimulation** if it is both a σ-relation and an opsimulation.

6.3. Theorem (Universal characterisation of satisfaction). *Given a transition system $\underline{S}$ over the set of labels L, the satisfaction relation $\models_S : S \not\rightarrow \Phi$ is the largest σ-opsimulation between $\underline{S}$ and $(\Phi, \{\langle\alpha\rangle^*\}_{\alpha \in L})$.*

Proof. By induction in the structure of the formula. The crucial point is that the satisfaction of a modal formula $\langle\alpha\rangle\phi$ is exactly the opsimulation condition for $\models_S$. □

The above theorem does exhibit an *intrinsic* relationship between opsimulation and modal formulae. A practical consequence of the result is a *proof technique* for showing that a state satisfies a formula, namely, find a σ-opsimulation relating them. We apply this technique in the following corollary:

6.4. Corollary.
Consider transition systems $\underline{S}$ and $\underline{T}$ and an opsimulation $\rho: S \not\rightarrow T$ between them. If $s(\rho)t$ and $t \models_{\underline{T}} \phi$ (for a formula ϕ), then $s \models_{\underline{S}} \phi$.

Of course, this corollary is rather well-known. The point here is to bring out its simple algebraic nature. The proof of the corollary hinges on the following lemma:

6.5. Lemma (σ-relations closed under precomposition).
Given a σ-relation $\rho: S \not\rightarrow F$ and a relation $R: T \not\rightarrow S$, the composite $R \bullet \rho: T \not\rightarrow F$ is a σ-relation.

The above lemma means that the poset of σ-relations $\sigma - \mathcal{R}el(S, F)$ is acted on the left by precomposition of relations. This observation and our reformulation of (op)simulations in §5 were the basis of our proposal for an abstract framework for simulations in terms of bimodules over bicategories of relations [9]. We are presently limiting ourselves to reformulating the theory in the classical setting (relations on $\mathcal{S}et$) to extract the main concepts and techniques necessary to organise such an axiomatic framework. We conclude our analysis in the following subsection by showing the role of the biclosed structure of $\mathcal{R}el$ in this theory.

6.1. Biclosed structure of $\mathcal{R}el$ and the completeness of observational opsimilarity

We have seen in Corollary 6.4 that two states s and t related by an opsimulation are *observationally opsimilar*, *i.e.* any observation (modal formula) satisfied by t is also satisfied by s.

Let us now consider the converse of the above corollary, that is, whether given transition systems $\underline{S}$ and $\underline{T}$ and states $s \in S$ and $t \in T$ which are observationally opsimilar:

$$\forall \phi \in \Phi.(t \models_{\underline{T}} \phi) \Longrightarrow (s \models_{\underline{S}} \phi)$$

implies that s opsimulates t (*i.e.* that there is an opsimulation relating them). The usual approaches to this problem (*cf.*[7]) impose restrictions on the systems so that the above condition gives an opsimulation between $\underline{S}$ and $\underline{T}$.

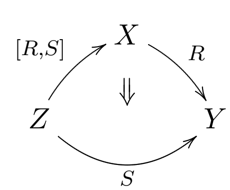
Let us review the biclosed structure of the bicategory $\mathcal{R}el$. Given a relation $R: X \not\rightarrow Y$, postcomposition with R

$$(\_) \bullet R : \mathcal{R}el(Z, X) \to \mathcal{R}el(Z, Y)$$

has a right adjoint $[R, \_] : \mathcal{R}el(Z, Y) \to \mathcal{R}el(Z, X)$ (right closed structure) given as follows: for a relation $S: Z \not\rightarrow Y$

$$[R, S](z, x) \equiv \forall y \in Y. x(R)y \Longrightarrow z(S)y$$

so that we have a **lifting** diagram



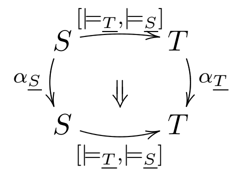
with the property that for any relation $T: Z \not\rightarrow X$

$$(T \bullet R \subseteq S) \quad \text{iff} \quad T \subseteq [R, S]$$

6.6. Remark. Since $\mathcal{R}el$ is self-dual (via ^{o}), its left closed structure (right adjoint to precomposition with R) is obtained from its right one:

$$\langle R, \_\rangle \equiv [R^o, (\_)^o]^o$$

We are now concerned with the relation $[\models_{\underline{T}}, \models_{\underline{S}}]\colon S \not\rightarrow T$. Any opsimulation between S and T is contained in it, so let us see what conditions are required for it to be an opsimulation: for a label α we would demand



In view of Proposition 6.1 and the universal property of $[\models_{\underline{T}}, \models_{\underline{S}}]$, it would suffice[3] that precomposition with $\alpha_{\underline{S}}\colon S \not\rightarrow S$ preserve the lifting defining $[\models_{\underline{T}}, \models_{\underline{S}}]$, that is

$$\alpha_{\underline{S}} \bullet [\models_{\underline{T}}, \models_{\underline{S}}] \cong [\models_{\underline{T}}, \alpha_{\underline{S}} \bullet \models_{\underline{S}}]$$

which boils down to

$$\forall \phi \in \Phi.\, t \models_{\underline{T}} \phi \Longrightarrow (\exists s' \in S.\, s(\alpha_{\underline{S}})s' \wedge s' \models_{\underline{S}} \phi)$$

implies

$$\exists s' \in S.\, s(\alpha_{\underline{S}})s' \wedge (\forall \phi \in \Phi.\, t \models_{\underline{T}} \phi \Longrightarrow s' \models_{\underline{S}} \phi)$$

To rephrase matters in more conventional model-theoretic terminology, when $\underline{S} = \underline{T}$, let $\Gamma_t = \{\phi \mid t \models_{\underline{T}} \phi\}$. The above condition amounts to:

If Γ_t is finitely satisfiable in some α-succesor of s, then Γ_t is satisfiable in some α-successor of s.

This is the condition of the transition system $\underline{S}$ being *m-saturated* in the sense of Visser, which is equivalent to the notion of $\mathcal{H}$*-saturation* introduced therein, *cf.* [7]. This is essentially the most general hypothesis to ensure that observational similarity implies the existence of an opsimulation. What we have added here is the reorganisation of this proof in terms of universal properties, which sheds light into the categorical requirements of a framework to do *abstract simulation theory*.

7. Conclusions and related work

In [8] we proposed a categorical conceptualisation of the notion of *logical predicate* (via a structural analysis of 2-categories of fibrations [10]). Such analysis led to a precise formulation of the role of logical predicates in (co)induction principles for data structures [12]. The present work, initiated in [9], arose from our intention of casting simulations in the same algebraic-categorical framework (bisimulations being "congruences for the dynamic of a system"). Here we have

[3]The defining lifting for $[\models_{\underline{T}}, \models_{\underline{S}}]$ combined with the isomorphisms which exhibit $\models_S$ and $\models_T$ as opsimulations yield a (pasting) 2-cell which induces the required 2-cell in the diagram.

given a neat categorical analysis of the predicate logic interpretation of the attendant relational modalities (based on our universal characterisation of bicategories of relations). Furthermore, our categorical interpretation of modalities as a pseudo-functor from $\mathcal{R}el$ combined with a monoidal view of transition systems ($\sigma_{\underline{S}} : L^* \to \mathcal{R}el$), led us to establish an algebraic characterisation of modal satisfaction (Theorem 6.3), thereby exhibiting an *intrinsic relationship* with the notion of simulation. We have thus exposed latent algebraic structure enabling us to relate simulation/modal satisfaction in a sound and complete manner recovering known results with new proof techniques. We hope this analysis would prove valuable in the continuing development of program logics. As for related work, the referees pointed out to us that of [1] (which we already commented upon in Remark 3.2) and [13]. This latter considers a category whose objects are relations acting upon topological spaces (as a particular kind of frame) and goes on to set-up a logical system which can interpreted internally in the resulting quasi-topos. The relationship of this system to standard (either classical or constructive) modal logic is not altogether evident, so we cannot make a precise connection to our work presently.

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