A roadmap to the unification of weak categorical structures: transformations and equivalences among the various notions of pseudo-algebra

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1 The pseudo-monoidality quartet

Consider the following different (but equivalent!) presentations of the notion *monoidal category*[1]:

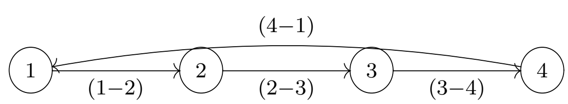
1. **algebraic**: psuedo-algebra for the *strict monoid monad* $M : \mathcal{C}at \to \mathcal{C}at$, $M\mathbb{C} = \Sigma_n \mathbb{C}^n$.

2. **universal fillers**: representable multicategory

3. **Kan complex/quasi-cat**: covariantly fibrant multicategory[2]

4. **Segalic**: pseudo-monoid in $\mathcal{C}at$, $m : \Delta \to \mathcal{C}at$, where Δ is the (*strict*) monoid classifier (finite ordinals and monotone maps) and m is a *pseudo*-functor which is strong monoidal with respect to the cartesian monoidal structure on $\mathcal{C}at$.

The notions in (2) and (3) where introduced in [Her00] and [Her03][3], respectively. These articles and specially [Her01] develop the relevant machinery to prove the (ordered!) cycle of equivalences (which restrict to their *strict* variants):



as follows:

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[1] This amounts to the case $n = 2$ of correspondences among weak n-categories.

[2] Although we use 'model structure' terminology to appeal to the reader's intuition, the fibrational notions involved are the proper categorical ones, whereby liftings are required to be universal.

[3] All references available from the author's homepage.

(1-2) [Her00, Th.9.8][4]

(2-3) [Her03, Th.4.1]

(3-4) This equivalence is missing from the (abridged) version [Her03] so we outline it here: since a covariant fibration of multicategories $p : M \to N$ induces one of (their freely associated) monoidal categories $Fp : FM \to FN$ (Th.2.4 in *ibid.*), they correspond to certain pseudo-functors $\mathcal{F}_p : FM \to \mathcal{C}at$. Since FM is monoidal and $\mathcal{C}at$ is cartesian monoidal, we may consider strong-monoidal pseudo-functors between them. We obtain an equivalence of 2-categories:

$$CovFib(\mathcal{M}ulticat)/M \simeq \mathcal{M}on\mathcal{C}at_{\otimes\mapsto\times}(FM, \mathcal{C}at)$$

For $M = \mathbf{1} = U\mathbf{1}$ the terminal multicategory (underlying the terminal monoidal category), our classification of lax morphisms ([Her01, Th.6.1]) indicates that $FU\mathbf{1}$ classifies monoids, and Th.9.2 of *ibid.* deduces from our explicit constructions of F and U that $FU\mathbf{1} \cong \Delta$.

(4-1) This direction is classical: restricting attention to the strict objects, we seek a left adjoint to the forgetful $Strict\mathcal{M}on\mathcal{C}at_{\otimes\mapsto\times}(\Delta, \mathcal{C}at) \to \mathcal{C}at$ (evaluation at 1), which recovers our original monoid monad M.

Even if we were to omit the new notions we have introduced (representable multicategories and covariant fibrations among them), there seems to be no other method in the literature to produce the transformation (1-4).

2 Three equivalent notions of pseudo-algebra

The appropriate level of abstraction for the above correspondences starts with a 2-monad[5] $\mathsf{T} : \mathcal{K} \to \mathcal{K}$ where

- the 2-category $\mathcal{K}$ admits a calculus of bimodules (*i.e.* pullbacks of fibrations and cofibrations, pullback stable coidentifiers and Kleisli objects for monads on bimodules.

[4] We notice that the adjoint characterisation of representability for a multicategory was only discovered in *ibid.* This required the development of the 2-category of multicategories with its remarkable adjunctions, previously missing in the literature.

[5] We adopt Kelly's method of studying weak structure based on *strict* notions. There are relevant general results to justify this approach, but we will not indulge on them here.

- the 2-monad T is cartesian, and its 2-functor preserves bimodules, their composites and identities, and their Kleisli objects.

the above references develop the relevant *transformational machinery* to set up the equivalences:

$$\text{Ps-T-}\mathcal{A}lg \overset{ps-rep}{\quad\quad} \text{Rep}-\text{Lax-Bimod(T)-alg} \overset{rep-fib}{\quad\quad} CovFib(\text{Lax-Bimod(T)-alg})$$

where

- the 2-category Lax-Bimod(T)-alg consists of normal lax algebras for the pseudo-monad induced by T on Bimod($\mathcal{K}$). These are *abstract multicategories*[6].
- The qualificative Rep$-$ indicates the abstract multicategories which admit a pseudo-algebra structure (necessarily a left adjoint to the unit of the lax-idempotent monadic adjunction of T-$\mathcal{A}lg$ over Lax-Bimod(T)-alg *cf.*[Her01, Prop.5.3]).
- The theory of (covariant) fibrations for abstract multicategories is developed in [Her03]. The adjective *CovFib* refers to those abstract multicategories such that the unique morphism to the terminal multicategory is a covariant fibration.

The transition equivalences are realised as follows:

$ps - rep$ [Her01, Th.5.6]. We should highlight the intrinsic characterisation of adjoint pseudo-algebras on abstract multicategories in terms of representability of the structure bimodule (Th.5.4 of *ibid.*). This characterisation amounts to the presence of 'universal filler' cells in the abstract setting, closed under 'multicategory' composition.

$rep - fib$ [Her03, Th.4.1]. We should point that the techniques required to prove this abstract version are wholly different from what can be done in the concrete case of $\mathcal{S}et$-based ordinary multicategories; we appeal essentially to 2-fibrational arguments.

The missing 'Segalic' version of structure-preserving pseudo-functors out of the 'lax T-object classifier' $FU\mathbf{1}$ can be implemented assuming our ambient 2-category $\mathcal{K}$ admits a classifier of (representable) fibrations which can

[6]Most of our theory goes through for the case of *symmetric/braided* multicategories, but some of the free constructions require a little more subtlety. The results still hold

be endowed with a pseudo-T-algebra structure, as it happens with $\mathcal{C}at$ and its cartesian monoidal structure.

Let us conclude by emphasizing that our theory develops a machinery to *produce* the relevant notions functorially out of the given 2-monad T, with the attendant equivalences.

3 Coherence: why universality matters

We conclude this quick tour of the land of pseudo-algebras with a simple coherence theorem for algebraic structures characterised by universal properties, *i.e.* adjoint pseudo-algebras for a lax idempotent 2-monad (Kock-Zöberlein property). Its proof will be given elsewhere, but we point out that it follows easily from the analysis in [Her01, §7]. We write $\mathcal{T}$-alg for the 2-category of strict $\mathcal{T}$-algebras and strict morphisms between them, and Ps-$\mathcal{T}$-alg for pseudo-algebras and strong morphisms.

3.1. Theorem. *Let $\mathcal{K}$ be a 2-category which admits Eilenberg-Moore objects for idempotent monads in it, and let $\mathcal{T}$ be a lax idempotent 2-monad on $\mathcal{K}$. The inclusion $\iota : \mathcal{T}\text{-alg} \to \text{Ps-}\mathcal{T}\text{-alg}$ admits a left biadjoint whose unit is an equivalence. Consequently ι is a biequivalence of 2-categories.*

3.2. Remark. The above statement is based on the observation that Kleisli and Eilenberg-Moore objects are equivalent for idempotent monads. However, we migh still need some exactness conditions on $\mathcal{T}$ to make the argument go through.

References

[Her00] C. Hermida. Representable multicategories. *Advances in Mathematics*, 151:164–225, 2000. Available at `http://www.cs.math.ist.utl.pt/s84.www/cs/claudio.html`.

[Her01] C. Hermida. From coherent structures to universal properties. *Journal of Pure and Applied Algebra*, 165(1):7–61, 2001. preprint available math.CT/0006161.

[Her03] C. Hermida. Fibrations for abstract multicategories. In *Proceedings of Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories, Fields Institute, Toronto, September 23-28,2002*. AMS, 2003. in press.