A Fibrational Framework for Possible-World Semantics of ALGOL*-like Languages*

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1 Introduction

Pioneering work by John Reynolds and Frank Oles [Rey81a, Ole82, Ole85, Ole97, OT92] showed how block-structured storage management in ALGOL-like languages [OT97] may be explicated using a semantics based on functor categories $\mathbf{W} \Rightarrow \boldsymbol{\mathcal{S}}$, where $\mathbf{W}$ is a suitable category of "worlds" characterizing local aspects of storage structure, and $\boldsymbol{\mathcal{S}}$ is a conventional semantic category of sets or domains. Every programming-language type θ is interpreted as a functor $[\![\theta]\!] \colon \mathbf{W} \to \boldsymbol{\mathcal{S}}$ and every programming-language term-in-context $\pi \vdash X \colon \theta$ is interpreted as a natural transformation $[\![\pi \vdash X \colon \theta]\!] \colon [\![\pi]\!] \stackrel{\cdot}{\to} [\![\theta]\!]$.

This functor-category framework was later exploited to analyze the non-interference predicate in Reynolds's specification logic [Rey81b, Ten90, O'H90, OT93a], block expressions in ALGOL-like languages [Ten85], and the concept of passivity in a variant of Reynolds's Syntactic Control of Interference [Rey78, OP⁺99].

O'Hearn and Tennent [OT93b, OT95] obtained a more precise analysis of block structure by internalizing additional uniformity contraints inspired by Reynolds's relational parametricity [Rey83]. This work also uses structures of the form $\mathbf{W} \Rightarrow \boldsymbol{\mathcal{S}}$ but $\mathbf{W}$ and $\boldsymbol{\mathcal{S}}$ are now reflexive graphs, with appropriate binary-relational categories above the usual categories of worlds and of sets (or domains). This framework was developed further by Reddy [Red97] and Dunphy [Dun02] by imposing additional conditions on $\mathbf{W}$ and $\boldsymbol{\mathcal{S}}$.

O'Hearn and Reynolds [OR00] describe an alternative approach to the semantics of local storage: the source language is translated into a polymorphic linear lambda calculus, which is then interpreted using a semantics with relational parametricity constraints. For example, $(\theta_0 \to \theta_1)^*(\alpha)$, the translation of type $\theta_0 \to \theta_1$ in world α, is defined to be $\forall \beta .\, \theta_0^*(\alpha \otimes \beta) \to \theta_1^*(\alpha \otimes \beta)$; here, β may be thought of as the "new" storage allocated between the definition of the procedure and an application. Possible worlds (states) are thus modelled by tensor products of free type variables, and phrase types (as in the example above) are coded so as to be meaningful on extensions of the state (a further tensoring of free type variables), allowing for the possibility that a procedure is invoked in an expanded state (extra variables) from that in which it is defined.

Here, we use the categorical concept of *fibration* (or *fibered category*) to provide a general framework within which it should be possible to express and compare these approaches to semantics. [Jac99] provides a fairly comprehensive account of fibrations in categorical logic and type theory.

Fibrations are relevant here for three reasons: first, as models of *indexing* by worlds; second, as categories of "relations" above categories, as in Hermida's analysis of logical relations [Her93] above cartesian closed categories; and, third, as models of polymorphic languages (i.e., languages with type variables). All of these are standard applications of fibrations, discussed in, for example, [Jac99]. When these uses of fibrations are *combined*, one obtains fibrations *of* fibrations, that is to say, fibrations in the 2-category $\mathcal{F}\!\mathit{ib}$ of fibered categories, a sub-2-category of the arrow 2-category $\mathcal{C}\mathit{at}^{\rightarrow}$. In general, a morphism $p\colon E \longrightarrow B$ in any 2-category $\mathcal{K}$ is said to be a *fibration in* $\mathcal{K}$ if, for every object X, the functor $\mathcal{K}(X,p)\colon \mathcal{K}(X,E) \longrightarrow \mathcal{K}(X,B)$ is a fibration of categories.

The relevant theory of fibrations over a fibration is developed in [Hera], where the purpose is to provide a framework for logical systems over polymorphic type theories [Herb]. For example, consider Plotkin and Abadi's *logic of parametric polymorphism* [PA93]. A proposition φ will in general be in the scope of quantifications with respect to *type* variables X, *individual* variables $x\colon A$, and *predicate* (or *relation*) variables $R \subset B$ (or $R \subset B \times C$), where the type expressions $A, B, C, \ldots$ may involve type variables. The judgements for proposition syntax may be conveniently presented in the following two-dimensional form:

$$\begin{array}{c|c} \Psi & \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array}$$

where

- $\Gamma = \ldots, X_i, \ldots$ is a *kind context* of type variables $\ldots, X_i, \ldots$;
- $\Theta = \ldots, x_i\colon A_i, \ldots$ is a *type context* of individual variables x_i of respective types $\Gamma \vdash A_i\ \mathit{Type}$ in kind context Γ; and
- $\Psi = \ldots, R_i \subset A_i, \ldots$ is a *proposition-kind context* of predicate (or relation) variables R_i on respective types $\Gamma \vdash A_i\ \mathit{Type}$ in kind context Γ.

The syntax rules for the three forms of quantification are then

$$\frac{\begin{array}{c|c} \Psi & \varphi\ \mathit{Prop} \\ \hline \Gamma, X & \Theta \end{array}}{\begin{array}{c|c} \Psi & \forall X.\, \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array}} \qquad \frac{\begin{array}{c|c} \Psi & \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta, x\colon A \end{array}}{\begin{array}{c|c} \Psi & \forall x\colon A.\, \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array}} \qquad \frac{\begin{array}{c|c} \Psi, R \subset B & \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array}}{\begin{array}{c|c} \Psi & \forall R \subset B.\, \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array}}$$

Categorically, such a logic would be modelled by a commuting diagram of fibrations

$$\begin{array}{ccc} \mathbf{Prop} & \longrightarrow & \mathbf{Type} \\ \big\downarrow & & \big\downarrow \\ \mathbf{PropKind} & \longrightarrow & \mathbf{Kind} \end{array} \qquad\qquad \begin{array}{ccc} \begin{array}{c|c} \Psi & \varphi\ \mathit{Prop} \\ \hline \Gamma & \Theta \end{array} & \longmapsto & \Gamma \vdash \Theta \\ \big\downarrow & & \big\downarrow \\ \Gamma \vdash \Psi & \longmapsto & \Gamma \end{array}$$

which would actually be a fibration of fibrations; precise definitions are given in Appendix A. We will here construct a similar fibrational structure for "parametric" functor category semantics of Algol.

Recapping, the sources for this work are threefold:

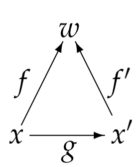
- the functor-category approach to semantics of Algol-like languages enhanced with reflexive graphs of relations to impose parametricity constraints [OT93b, OT95];
- the interpretation of Algol in polymorphic linear lambda calculus [OR00], which motivates our construction in Section 2 of a fibration from a functor category;
- the treatment of relational polymorphism based on fibrations in $\mathcal{F}\boldsymbol{ib}$ [Hera], which allows us to extend this construction to the construction of a fibration over a fibration from a reflexive graph over functor categories, thereby bringing the relational framework over functor categories into the realm of fibered categorical type theory.

2 From Functor Categories to Fibrations Using Slices

We begin by showing how, from arbitrary categories $\mathbf{W}$ and $\boldsymbol{\mathcal{S}}$, we may obtain a fibration on $\mathbf{W}$ whose fibers are functor categories; in particular, if $\mathbf{W}$ has a terminal object, the fiber over it is the functor category $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$. Note that henceforth we will consistently work with *contravariant* functors on worlds; that is, we consider $\mathbf{W}$ to be a small category whose morphisms are, typically, projections, rather than "expansions." This is the opposite of the convention established by Reynolds and Oles, but fits better with mathematical practice. In categorical logic, re-indexing contravariantly along projections corresponds to *weakening* and quantifiers are explained as adjoints to weakening functors.

A standard way to construct a fibration on a category $\mathbf{W}$ is to define categories indexed on $\mathbf{W}$, that is, to define a (pseudo) functor S from $\mathbf{W}^{\mathrm{op}}$ to the category $\mathcal{C}\boldsymbol{at}$ of (locally small) categories, and apply the Grothendieck construction to it; in fact, *every* fibration arises in this way.

In our case, given categories $\mathbf{W}$ and $\boldsymbol{\mathcal{S}}$, we define the functor $S\colon \mathbf{W}^{\mathrm{op}} \longrightarrow \mathcal{C}\boldsymbol{at}$ as follows: for any world w, the relevant fiber category $S(w)$ will be the category $(\mathbf{W}/w)^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ of all contravariant functors from $\mathbf{W}/w$ to $\boldsymbol{\mathcal{S}}$, where the slice category $\mathbf{W}/w$ has as objects all $\mathbf{W}$-morphisms into w and, as morphisms from $f\colon x \longrightarrow w$ to $f'\colon x' \longrightarrow w$, all commuting diagrams of the form



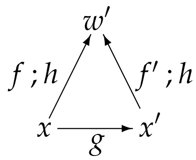
This construction retains in $S(w)$ information about the behaviour of any functor $F\colon \mathbf{W}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}}$ in possible future worlds derived from w, bearing in mind that we think of a morphism $f\colon x \longrightarrow w$ as a projection from an expanded world x to w, the contravariant action of F on f being a "logical weakening" of the object to the expanded context. This is consistent with the philosophy behind possible-world semantics. In fact, from the perspective of world w, all that matters about a functor F is its behaviour in the "restricted" universe $\mathbf{W}/w$.

For functors $(\mathbf{W}/w)^{\mathrm{op}} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{\mathcal{S}}$, the morphisms from F to G are, of course, the natural transformations $(\mathbf{W}/w)^{\mathrm{op}} \underset{G}{\overset{F}{\Downarrow \eta}} \boldsymbol{\mathcal{S}}$. Note that, if $\mathbf{W}$ has a terminal object 1, $S(1)$ is just the familiar functor category $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$.

Proposition 1 *For any small category* $\mathbf{W}$*, if* $\boldsymbol{\mathcal{S}}$ *is cartesian closed and complete, so is* $S(w)$ *for every* $w \in \mathbf{W}$.

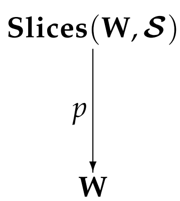
Proof. See Appendix B.1. ■

To complete the definition of S, we must define, for every $h\colon w \to w'$, a re-indexing functor $S(h)\colon S(w') \to S(w)$. Note that any $h\colon w \to w'$ induces by composition a functor $\Sigma_h\colon (\mathbf{W}/w) \longrightarrow (\mathbf{W}/w')$, taking the diagram above to



where ; denotes composition in diagrammatic order. So, for any $F\colon (\mathbf{W}/w')^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}}$ in $S(w')$, we define $S(h)(F)$ in $S(w)$ to be $\Sigma_h^{\mathrm{op}}\,; F$. Similarly, for any morphism $\eta\colon F \dot{\to} G$ in $S(w')$, $S(h)(\eta)$ in $S(w)$ is $\Sigma_h^{\mathrm{op}}\,; \eta$, so that, for any $f\colon x \to w$, $S(h)(\eta)(f) = \eta(\Sigma_h^{\mathrm{op}}(f)) = \eta(f\,; h)$. In short, the functorial action for h is simply precomposition with the functor Σ_h induced by h.

These definitions make S a functor from $\mathbf{W}^{\mathrm{op}}$ to $\boldsymbol{\mathcal{C}at}$; we may then use the Grothendieck construction to obtain a split fibration on $\mathbf{W}$, which we portray as follows:



2.1 Cartesian Closure and Completeness

Proposition 2 *If* $\boldsymbol{\mathcal{S}}$ *is complete and cartesian closed, the fibration* $p\colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \to \mathbf{W}$ *is fibrewise cartesian closed and this structure is preserved by re-indexing.*

Proof. See Appendix B.2. ■

Proposition 3 *If* $\boldsymbol{\mathcal{S}}$ *is complete, the fibration* $p\colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \to \mathbf{W}$ *admits products; dually, if* $\boldsymbol{\mathcal{S}}$ *is co-complete, it admits co-products.*

Proof. See Appendix B.3. ■

2.2 Recovering the Functor Category from the Fibration

It was mentioned previously that if $\mathbf{W}$ has a terminal object 1, the functor category $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ is equivalent to the fiber over 1. More generally (such as when $\mathbf{W}$ is only symmetric monoidal),

Proposition 4 *For any* $\mathbf{W}$*, the functor category* $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ *is equivalent to the category of* cartesian sections *of* $p\colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \mathbf{W}$*; that is, all functors* $s\colon \mathbf{W} \longrightarrow \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})$ *such that* $s\,;p = \mathrm{id}_{\mathbf{W}}$ *and, for every* $\mathbf{W}$*-morphism* f*,* $s(f)$ *is cartesian.*

Proof. See Appendix B.4. ■

3 Relations

We now consider how to impose relation-preservation constraints on objects and morphisms analogous to the "parametric" functors and natural transformations of O'Hearn and Tennent [OT93b, OT95].

Suppose we have two categories $\mathbf{RW}$ and $\boldsymbol{\mathcal{RS}}$ with functors $rw\colon \mathbf{RW} \longrightarrow \mathbf{W} \times \mathbf{W}$ and $rs\colon \boldsymbol{\mathcal{RS}} \longrightarrow \boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}}$. An object W of $\mathbf{RW}$ is regarded as being a kind of abstract relation on $(w_0, w_1) = rw(W)$; similarly, an object S of $\boldsymbol{\mathcal{RS}}$ is typically a binary relation between sets or domains s_0 and s_1, where $(s_0, s_1) = rs(S)$. Morphisms in $\mathbf{RW}$ and $\boldsymbol{\mathcal{RS}}$ may be thought of as morphisms in $\mathbf{W} \times \mathbf{W}$ or $\boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}}$ that preserve these relations; see Example 4.1 in Section 4.

We assume the functors rw and rs are fibrations; but note that such fibrations may also be viewed as categorical spans:

$$\mathbf{W} \xleftarrow{\;rw\,;\,\pi_0\;} \mathbf{RW} \xrightarrow{\;rw\,;\,\pi_1\;} \mathbf{W} \qquad\qquad \boldsymbol{\mathcal{S}} \xleftarrow{\;rs\,;\,\pi_0\;} \boldsymbol{\mathcal{RS}} \xrightarrow{\;rs\,;\,\pi_1\;} \boldsymbol{\mathcal{S}}$$

where π_0 and π_1 are the projections. The exponentiation $rw^{\mathrm{op}} \Rightarrow rs$ in the 2-category of spans is then similar to the reflexive-graph exponentiation used by O'Hearn and Tennent. But we want the exponentiation to be "fibered over worlds," as in Section 2. This will involve defining a category $\mathbf{Slices}(rw, rs)$ fibered over $\mathbf{RW}$ by construction, but *also* fibered over $\mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \times \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})$, with commutativity as follows:

$$\begin{array}{ccc}
\mathbf{Slices}(rw, rs) & \xrightarrow{\;\widetilde{rw}\;} & \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \times \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \times p} \\
\mathbf{RW} & \xrightarrow[\;rw\;]{} & \mathbf{W} \times \mathbf{W}
\end{array}$$

Then $(q, p \times p)\colon \widetilde{rw} \longrightarrow rw$ is a fibration in the 2-category $\boldsymbol{\mathcal{F}ib}$ of fibrations.

Consider any "relation" W in $\mathbf{RW}$ with $rw(W) = (w_0, w_1)$. We define the fiber over W as follows:

- objects are triples $\big(\widetilde{F}, F_0, F_1\big)$ of functors such that

$$
\begin{array}{ccc}
(\mathbf{RW}/W)^{\mathrm{op}} & \xrightarrow{\quad\widetilde{F}\quad} & \mathcal{RS} \\
{\scriptstyle (rw/W)^{\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
(\mathbf{W}/w_0)^{\mathrm{op}} \times (\mathbf{W}/w_1)^{\mathrm{op}} & \xrightarrow[\quad F_0 \times F_1\quad]{} & \mathcal{S} \times \mathcal{S}
\end{array}
$$

  commutes, where rw/W is the functor obtained from rw by applying rw to objects and morphisms of $\mathbf{RW}/W$;

- morphisms from $\big(\widetilde{F}, F_0, F_1\big)$ to $\big(\widetilde{G}, G_0, G_1\big)$ are triples $\big(\widetilde{\eta}, \eta_0, \eta_1\big)$ of natural transformations:

$$
\begin{array}{ccc}
(\mathbf{RW}/W)^{\mathrm{op}} & \overset{\widetilde{F}}{\underset{\widetilde{G}}{\rightrightarrows}}\ {\scriptstyle \Downarrow \widetilde{\eta}} & \mathcal{RS} \\
{\scriptstyle (rw/W)^{\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
(\mathbf{W}/w_0)^{\mathrm{op}} \times (\mathbf{W}/w_1)^{\mathrm{op}} & \overset{F_0 \times F_1}{\underset{G_0 \times G_1}{\rightrightarrows}}\ {\scriptstyle \Downarrow \eta_0 \times \eta_1} & \mathcal{S} \times \mathcal{S}
\end{array}
$$

In particular, if 1 is a terminal object in **RW**, the fiber over 1 is just $rw^{\mathrm{op}} \Rightarrow rs$, with objects and morphisms as follows:

$$
\begin{array}{ccc}
\mathbf{RW}^{\mathrm{op}} & \overset{\widetilde{F}}{\underset{\widetilde{G}}{\rightrightarrows}}\ {\scriptstyle \Downarrow \widetilde{\eta}} & \mathcal{RS} \\
{\scriptstyle rw^{\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
\mathbf{W}^{\mathrm{op}} \times \mathbf{W}^{\mathrm{op}} & \overset{F_0 \times F_1}{\underset{G_0 \times G_1}{\rightrightarrows}}\ {\scriptstyle \Downarrow \eta_0 \times \eta_1} & \mathcal{S} \times \mathcal{S}
\end{array}
$$

which corresponds, roughly, to the category of "parametric" functors and natural transformations of [OT93b, OT95], but with the relational constraints imposed in the fibers, rather than in the base.

To define the re-indexing functors, consider any map $\widetilde{h}\colon W \longrightarrow W'$ in **RW** and suppose $rw\big(\widetilde{h}\big) = (w_0, w_1) \xrightarrow{(h_0,h_1)} (w'_0, w'_1)$; then any object $\big(\widetilde{G}, G_0, G_1\big)$ of the fiber on W' is mapped to a triple of functors obtained by pre-composition with Σ functors induced by $\widetilde{h}$, h_0, and h_1, respectively, as follows:

$$
\begin{array}{ccccc}
(\mathbf{RW}/W)^{\mathrm{op}} & \xrightarrow{\quad\Sigma^{\mathrm{op}}_{\widetilde{h}}\quad} & (\mathbf{RW}/W')^{\mathrm{op}} & \xrightarrow{\quad\widetilde{G}\quad} & \mathcal{RS} \\
{\scriptstyle (rw/W)^{\mathrm{op}}}\big\downarrow & & {\scriptstyle (rw/W')^{\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
(\mathbf{W}/w_0)^{\mathrm{op}} \times (\mathbf{W}/w_1)^{\mathrm{op}} & \xrightarrow[\Sigma^{\mathrm{op}}_{h_0} \times \Sigma^{\mathrm{op}}_{h_1}]{} & (\mathbf{W}/w'_0)^{\mathrm{op}} \times (\mathbf{W}/w'_1)^{\mathrm{op}} & \xrightarrow[\quad G_0 \times G_1\quad]{} & \mathcal{S} \times \mathcal{S}
\end{array}
$$

and similarly for fiber morphisms. This defines a functor from $\mathbf{RW}^{\mathrm{op}}$ to $\mathcal{C}\boldsymbol{at}$, and applying the Grothendieck construction to it gives the desired split fibration on $\mathbf{RW}$:

$$\begin{array}{c} \mathbf{Slices}(rw, rs) \\ \downarrow{\scriptstyle q} \\ \mathbf{RW} \end{array}$$

We may now define a functor $\widetilde{rw}\colon \mathbf{Slices}(rw, rs) \longrightarrow \mathbf{Slices}(\mathbf{W}, \mathcal{S}) \times \mathbf{Slices}(\mathbf{W}, \mathcal{S})$ as follows: an object $\left(\widetilde{F}, F_0, F_1\right)$ is mapped to (F_0, F_1), and similarly for morphisms. This gives us the following commuting diagram:

$$\begin{array}{ccc} \mathbf{Slices}(rw, rs) & \xrightarrow{\;\widetilde{rw}\;} & \mathbf{Slices}(\mathbf{W}, \mathcal{S}) \times \mathbf{Slices}(\mathbf{W}, \mathcal{S}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \times p} \\ \mathbf{RW} & \xrightarrow[\;rw\;]{} & \mathbf{W} \times \mathbf{W} \end{array}$$

This means that $(q, p \times p)\colon \widetilde{rw} \longrightarrow rw$ is a fibration in the 2-category of spans; furthermore,

Proposition 5 *In the preceding diagram, $(q, p \times p)\colon \widetilde{rw} \longrightarrow rw$ is a fibration in $\mathcal{F}\boldsymbol{ib}$ whenever p and rw are fibrations.*

Proof. See Appendix B.5. ■

As we will see in the next section, the examples of categories of worlds and relations in the literature all involve a functor rw which *is* a fibration; however, it may be worthwhile to note that it is actually the fibrational nature of rs that is of crucial importance.

Corollary 6 *If rs is a fibration, so is $rw^{\mathrm{op}} \Rightarrow rs$ for* any *functor rw.*

Proof. See Appendix B.6. ■

Proposition 7 *If rs is a fibration of complete cartesian closed categories, the preceding diagram is a morphism of fibered cartesian closed categories with products.*

Proof. See Appendix B.7. ■

Remark on Parametricity We would like to be able to demand a "logical-relations" lifting $\mathbf{Slices}(\mathbf{W}, \mathcal{S}) \longrightarrow \mathbf{Slices}(rw, rs)$ that

- preserves Cartesian-closed structure fiberwise,
- extends equality on "closed types" (i.e., constant functors), and
- satisfies "identity extension"; i.e., the action of the functor on relations preserves identity relations.

Such a lifting can be defined (by structural induction) on those functors that denote phrase types in the semantics of ALGOL; however, the associated relational action cannot be guaranteed to satisfy identity extension.

The usual solution is to replace the original model by its "parametric completion" [RR94, BM05], in this case, morphisms of reflexive graphs (as in [OT95], where they are shown to form a Cartesian closed category), whereby the interpretations of types come already equipped with a relational action satisfying identity extension. We would then consider "relations" between two such parametric functors to be a "relational action" on the underlying pair of ordinary functors.

4 Categories of Worlds and Relations

In this section, we present categories of worlds and relations that have been used in programming-language semantics as *fibrations*.

Another perspective on these categories is given in [HT], where it is shown that they can all be obtained by freely adjoining "monoidal indeterminates" to suitable symmetric monoidal (closed) categories, and hence have universality properties.

Example 4.1 The category of worlds introduced in [Ten90] may be described as follows.

- Objects are sets W, X, Y, $\ldots$; these are regarded as sets of local states.
- Morphisms from X to W are pairs $(V, X \overset{m}{\rightarrowtail} W \times V)$ where V is a set (of values for "new" local variables) and m is a monic function (to impose constraints on the local states).
- The identity on W is $\big(1, \langle \mathrm{id}_W, !_W \rangle\big)$ and the composite of $(V, m)\colon Y \longrightarrow X$ and $(V', m')\colon X \longrightarrow W$ is $\big(V' \times V, m \,;(m' \times \mathrm{id}_V)\big)$.

This description makes it clear that we can construct a category $\mathcal{T}(\mathbf{C})$ of "worlds" not only from a category $\mathbf{C}$ of sets and functions, but from *any* category with finite products; furthermore, any finite-product and mono preserving functor $rs\colon \mathbf{D} \longrightarrow \mathbf{C}$ between such categories induces a functor $\mathcal{T}(rs)\colon \mathcal{T}(\mathbf{D}) \longrightarrow \mathcal{T}(\mathbf{C})$ on the categories of worlds; that is, $\mathcal{T}$ is a *functor* from categories with finite products and finite-product and mono preserving functors between them to ***Cat***. This will allow us to construct a fibration on worlds as the functorial image of a familiar sub-object fibration on sets as follows.

Let

- **Set** be a small category of sets and all functions between them,
- **Rel** be the category whose objects are binary relations $R\colon W_0 \longleftrightarrow W_1$ on pairs of **Set**-objects and whose morphisms are relation-preserving pairs of functions:

  $$(f, g)\colon (R\colon W_0 \longleftrightarrow W_1) \longrightarrow (S\colon X_0 \longleftrightarrow X_1)$$

  such that $w_0[R]w_1$ implies $f w_0[S] g w_1$ and

- $r\colon \mathbf{Rel} \longrightarrow \mathbf{Set} \times \mathbf{Set}$ be the functor such that $r(R\colon W_0 \longleftrightarrow W_1) = (W_0, W_1)$, and similarly for morphisms.

The category **Rel** has products: given $R\colon W_0 \longleftrightarrow W_1$ and $S\colon X_0 \longleftrightarrow X_1$, their product is $R \times S\colon W_0 \times X_0 \longleftrightarrow W_1 \times X_1$ such that

$$(w_0, x_0)[R \times S](w_1, x_1) \text{ iff } (w_0[R]w_1 \text{ and } x_0[S]x_1)$$

Since monos in **Rel** are simply pairs of monomorphisms in **Set**, r preserves products and monos. So, if we let $\mathbf{W} = \mathcal{T}(\mathbf{Set})$ be our category of worlds (based on **Set**), $\mathbf{RW} = \mathcal{T}(\mathbf{Rel})$ is a category of relations above $\mathbf{W} \times \mathbf{W}$. In fact, an object of **RW** is a binary relation $R\colon W_0 \longleftrightarrow W_1$ and a morphism from $S\colon X_0 \longleftrightarrow X_1$ to $R\colon W_0 \longleftrightarrow W_1$ is a relation $T\colon V_0 \longleftrightarrow V_1$ together with monic functions $X_0 \overset{m_0}{\rightarrowtail} W_0 \times V_0$ and $X_1 \overset{m_1}{\rightarrowtail} W_1 \times V_1$ such that

$$\begin{array}{ccc} X_0 & \overset{m_0}{\rightarrowtail} & W_0 \times V_0 \\ S \updownarrow & \Rightarrow & \updownarrow R \times T \\ X_1 & \underset{m_1}{\rightarrowtail} & W_1 \times V_1 \end{array}$$

that is, if $x_0[S]x_1$ then $m_0(x_0)[R \times T]m_1(x_1)$.

Now $rw = \mathcal{T}(r)\colon \mathbf{RW} \longrightarrow \mathbf{W} \times \mathbf{W}$ is the forgetful functor that map relations to their domains and a morphism as above to the underlying **W**-morphisms. It is easily shown that rw is a fibration: for any **RW**-object $R\colon W_0 \longleftrightarrow W_1$ and $(\mathbf{W} \times \mathbf{W})$-morphism $\big((V_0,\, X_0 \overset{m_0}{\rightarrowtail} W_0 \times V_0), (V_1,\, X_1 \overset{m_1}{\rightarrowtail} W_1 \times V_1)\big)$, a cartesian lifting has as domain the relation $S\colon X_0 \longleftrightarrow X_1$ defined by $x_0[S]x_1$ iff $w_0[R]w_1$, where $m_0(x_0) = (w_0, v_0)$ and $m_1(x_1) = (w_1, v_1)$, and $v_0[T]v_1$ is true for all $v_0 \in V_0$ and $v_1 \in V_1$.

Let us also point out that $r\colon \mathbf{Rel} \longrightarrow \mathbf{Set} \times \mathbf{Set}$ admits an equality (the usual diagonal relation on a set). **Rel** is cartesian closed and (co-)complete and both r and $Eq\colon \mathbf{Set} \longrightarrow \mathbf{Rel}$ preserve this structure.

Example 4.2 The Oles category of worlds [Ole82, Ole85, Ole97] and the O'Hearn-Tennent category of relations above this [OT93b] are obtained by restricting the function components of the morphisms for Tennent's categories to isomorphisms, and so they are similarly the functorial image of a subobject fibration (an observation noted in [OT95, Section 10.1] and attributed there to Andy Pitts) and the relevant "domains" functor is again a fibration.

Example 4.3 The category of worlds originally proposed for interpreting ALGOL is described by Reynolds [Rey81a] using sets and partial functions. We rephrase his construction using total maps from an arbitrary cartesian closed category **C**.

The category $\mathcal{R}(\mathbf{C})$ has the same objects as **C**. A morphism $(g, G)\colon X \longrightarrow Y$ consists of a pair of **C**-morphisms $g\colon Y \longrightarrow X$ and $G\colon (X \Rightarrow X) \longrightarrow (Y \Rightarrow Y)$ satisfying the following:

1. G preserves composition and identities.

2. $G\,;(Y \Rightarrow g) = (g \Rightarrow X)$

3. $D_X\,;G = (g \Rightarrow G)\,;D_Y$, where for any object X, the diagonalisation morphism $D_X\colon X \Rightarrow (X \Rightarrow X) \longrightarrow (X \Rightarrow X)$ is defined by $D_X = \theta_X\,;(\delta_X \Rightarrow X)$, with $\theta_X\colon X \Rightarrow (X \Rightarrow X) \cong (X \times X \Rightarrow X)$ and $\delta_X\colon X \longrightarrow X \times X$.

Composition of morphisms is given componentwise, $(g, G)\,;(h, H) = (h\,;g, G\,;H)$, and the identity is $(\mathrm{id}, \mathrm{id})$. Intuitively, $g\colon Y \longrightarrow X$ projects out the small state embedded in a larger one, whereas $G\colon (X \Rightarrow X) \longrightarrow (Y \Rightarrow Y)$ maps any command on small states to the corresponding command on large states that preserves the values of new variables (Condition 2 above). The third condition is relevant to the object-oriented view of variables in Algol [Rey81a].

In fact, the category of possible worlds used by Oles [Ole82, Ole85] is isomorphic to the Reynolds category. For any category with finite products $\mathbf{C}$, the category $\mathcal{O}(\mathbf{C})$ has the same objects as $\mathbf{C}$, while a morphism $(g, \rho)\colon X \longrightarrow Y$ consists of a pair of $\mathbf{C}$-morphisms $g\colon Y \longrightarrow X$ and $\rho\colon X \times Y \longrightarrow Y$ satisfying the following:

1. $\rho\,;g = \pi_0$, with $\pi_0\colon X \times Y \longrightarrow X$ being the first projection

2. $\langle g, \mathrm{id}\rangle\,;\rho = \mathrm{id}$

3. $(X \times \rho)\,;\rho = \pi_{0,2}\,;\rho\colon X \times X \times Y \longrightarrow X \times Y$, where $\pi_{0,2}\colon X \times X \times Y \longrightarrow X \times Y$ selects the first and third components of the triple.

Composition of morphisms involves diagonalisation. Since we are interested in the case when $\mathbf{C}$ is cartesian closed, we present a simplified version using adjoint transposition. Given morphisms $(g, \rho)\colon X \longrightarrow Y$ and $(h, \rho')\colon Y \longrightarrow Z$, consider $\widehat{\rho}\colon X \longrightarrow (Y \Rightarrow Y)$, the adjoint transpose of ρ, and similarly for ρ'. The composite $(g, \rho)\,;(h, \rho')$ is $(h\,;g, \rho'')$ where ρ'' is the adjoint transpose of

$$\widehat{\rho}\,;\big((h\,;g) \Rightarrow \widehat{\rho'}\big)\,;D_Z\colon X \longrightarrow (Z \Rightarrow Z)$$

Intuitively, $g\colon Y \longrightarrow X$ is, again, a projection, and $\rho\colon X \times Y \longrightarrow Y$ replaces the X-part of a large state, leaving the values of the new variables invariant.

We may now describe the isomorphism between these categories of worlds. Any $\mathcal{R}(\mathbf{C})$-morphism (g, G) may be mapped to the $\mathcal{O}(\mathbf{C})$-morphism $\big(g, \rho_G\big)$ such that ρ_G applies G to a "constant" X-command that, for all input states, outputs the desired new state; more precisely, ρ_G is the adjoint transpose of $\kappa_X\,;G\colon X \longrightarrow (Y \Rightarrow Y)$, where for any object X, $\kappa_X\colon X \longrightarrow (X \Rightarrow X)$ is the adjoint transpose of the first projection $\pi_0\colon X \times X \longrightarrow X$. Intuitively, κ_X takes an element $x \in X$ to the constant x-valued function on X.

In the other direction, any $\mathcal{O}(\mathbf{C})$-morphism (g, ρ) may be mapped to the $\mathcal{R}(\mathbf{C})$-morphism (g, G_ρ) such that G_ρ uses g to project out the X-part of a Y-state, applies the relevant X-command to it, and then uses ρ to replace the X-part of the original state. In detail: $G_\rho = (g \Rightarrow \widehat{\rho})\,;D_Y$.

We leave to the reader the detailed calculations needed to verify that these constructions are mutually inverse.

Proposition 8 *For any cartesian closed category* $\mathbf{C}$*, the categories* $\mathcal{R}(\mathbf{C})$ *and* $\mathcal{O}(\mathbf{C})$ *are isomorphic.*

Because of this isomorphism, a category of relations fibered over the Reynolds category may be constructed as in Example 4.2 (or, more directly, by applying $\mathcal{R}$ to a subobject fibration on $\mathbf{C} \times \mathbf{C}$).

Example 4.4 Several authors [Mog90, OT92, OT93b, PS93, Sie96, Dun02] have used the category **Loc** of finite sets (of "locations") and injections (or inclusions) as a category of worlds. A systematic way of constructing a suitable category of relations fibered over $\mathbf{Loc} \times \mathbf{Loc}$ is to pull back the fibration $r\colon \mathbf{RW} \longrightarrow \mathbf{W} \times \mathbf{W}$ along the functor $J\colon \mathbf{Loc}^{\mathrm{op}} \longrightarrow \mathbf{W}$ that interprets a set $\{\ell_1, \ldots, \ell_n\}$ of locations as the cartesian product $S = V(\ell_1) \times \cdots \times V(\ell_n)$, where $V(\ell_k)$ is the set of values storable at location ℓ_k. An injection $i\colon \{\ell_1, \ldots, \ell_n\} \rightarrowtail \{\ell'_1, \ldots, \ell'_m\}$ is mapped to the pair (g, ρ), where $g\colon V(\ell'_1) \times \cdots \times V(\ell'_m) \longrightarrow V(\ell_1) \times \cdots \times V(\ell_n)$ projects out the components that are not in the image of i, and $\rho\colon S \times S' \longrightarrow S'$ substitutes the S-part of an S'-tuple, leaving the remaining components unchanged.

Example 4.5 The category of worlds and relations described by Dunphy [Dun02] is constructed less uniformly: the base category of worlds is the preorder of finite sets and inclusions, but the category of relations is defined by applying the Reynolds construction $\mathcal{R}(\cdot)$ to $r\colon \mathbf{Rel} \longrightarrow \mathbf{Set} \times \mathbf{Set}$. This is isomorphic to the dual of the fibration in Example 4.2 above, by Proposition 8. Pulling it back along the functor $J\colon \mathbf{Loc}^{\mathrm{op}} \longrightarrow \mathbf{W}$ yields a fibration on $\mathbf{Loc} \times \mathbf{Loc}$. To endow this fibration with an equality relation, Dunphy adds relations between state transformers $R_t\colon (W_0 \Rightarrow W_0) \longleftrightarrow (W_1 \Rightarrow W_1)$ satisfying axioms analogous to those for morphisms in $\mathcal{R}(\mathbf{Set})$.

Example 4.6 The main semantic innovation in [OR00] is the use of binary relations that may relate states to "undefined" states. We will show how these may be re-constructed in our framework.

First, note that the fibration $r\colon \mathbf{Rel} \longrightarrow \mathbf{Set} \times \mathbf{Set}$ may be obtained by the change-of-base construction from the fibration $c\colon \mathbf{Sub}(\mathbf{Set}) \longrightarrow \mathbf{Set}$ along the product functor $\times\colon \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$. In more detail: the category $\mathbf{Sub}(\mathbf{Set})$ has as its objects all subobjects $P \overset{m}{\rightarrowtail} X$ (thought of as predicates on X) and a morphism between $P \overset{m}{\rightarrowtail} X$ and $Q \overset{n}{\rightarrowtail} Y$ is a function $f\colon X \longrightarrow Y$ taking P to Q. Taking the codomains of subobjects yields a functor $c\colon \mathbf{Sub}(\mathbf{Set}) \longrightarrow \mathbf{Set}$. Clearly this construction can be performed on *any* category $\mathbf{C}$ in place of **Set**. The resulting functor $c\colon \mathbf{Sub}(\mathbf{C}) \longrightarrow \mathbf{C}$ is a fibration whenever $\mathbf{C}$ admits pullbacks of monos along arbitrary morphisms.

We next show that this construction yields a fibration when applied to certain categories of *partial* maps. When command executions may be non-terminating, the most natural model for command meanings is as *partial* functions on the relevant set of states. Let **Setp** be a small category of sets and all partial functions between them. It turns out that the objects of the category $\mathbf{Sub}(\mathbf{Setp})$ are the same as the objects of $\mathbf{Sub}(\mathbf{Set})$.[1] $\mathbf{Sub}(\mathbf{Setp})$ is fibered over **Setp**: the resulting re-indexing functor c^* for any partial function $c\colon S \rightharpoonup S'$ is given as the weakest (liberal) precondition:[2] $c^*(Q)(s)$ iff $Q\big(c(s)\big)$ whenever $c(s)$ is defined. The morphisms from P to Q are the valid Hoare triples $P\{c\}Q$.

A fibration of appropriate *binary* relations (on possibly distinct sets of states) may now be obtained by change of base along the product functor $\times$. Note that the *categorical* product $S \times S'$ in **Setp** is the set $S \oplus (S \otimes S') \oplus S'$, where $\oplus$ denotes disjoint union and $\otimes$ is the conventional cartesian product of sets. The projection $\pi_0\colon S \times S' \rightharpoonup S$ is defined by cases on $S \oplus (S \otimes S') \oplus S'$ as follows: $\pi_0(s) = s$ for $s \in S$, $\pi_0(s, s') = s$, and $\pi_0(s')$ is undefined for $s' \in S'$, and similarly for the other projection $\pi_1\colon S \times S' \rightharpoonup S'$. This construction yields binary relations which may be "preserved" by a pair (f, f') of partial functions, even if one is undefined on the relevant component of a related pair of arguments.

In practice, particularly when working with concrete examples, it is more convenient to make the "undefineds" explicit and work with the equivalent category $\mathbf{Set}_\perp$ of pointed sets and $\perp$-preserving *total* functions [Red97], but the objects are nonetheless *sets*, and not "flat domains." In particular, the analogous equivalence *fails* for categories of domains [Fio96]. In treating an Algol-like language, states become involved in elements of domains only as arguments or results of (possibly partial) *functions*. A set of partial functions (or $\perp$-preserving total functions) on sets may be ordered in the obvious way to form a domain. The following is a more accurate presentation of the type system used for the target language in [OR00]:

$$\begin{array}{lll} \sigma & ::= \ \alpha \mid I \mid \sigma \otimes \sigma & \text{Level 1} \\ A & ::= \ \sigma \multimap \sigma \mid A \to A \mid A \,\&\, A \mid \forall \alpha.\, A & \text{Level 2} \end{array}$$

where α ranges over variables for Level 1 types. The Level 1 types should denote sets and the Level 2 types should denote domains.

From such a category of binary relations on state sets, it is then possible to construct categories of worlds and relations on worlds in any of the ways discussed above. There do not appear to be any impediments to using the new relations on states in treating such features as passive expressions, non-interference predicates in a specification logic, and block expressions.

5 Discussion

In functor-category models, the semantic categories are typically cartesian closed and complete, but note that these properties are *not* required of the categories of worlds. Similarly, in the framework we have presented here, we have not tried to impose on $\mathbf{W}$ *all* of the properties of $\boldsymbol{\mathcal{S}}$. For example, the equality-relation functor $Eq\colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{RS}}$ gives

[1] A subobject in any bicategory of partial maps is always total, i.e., is the same as a subobject in the category of total maps.

[2] This property holds more generally for bicategories of partial maps $Ptl(\mathbf{C})$ when the subobject fibration $c\colon \mathbf{Sub}(\mathbf{C}) \to \mathbf{C}$ admits products along monomorphisms. See [Her02] for further analysis of fibrations over relations and partial maps.

$\mathcal{S}$ a reflexive-graph structure, but we have not required this for **W**. If a reflexive-graph structure is needed on worlds in order to incorporate parametricity, the relevant relations (including the distinguished equality) are taken from those on $\mathcal{S}$ via the "states" functor.

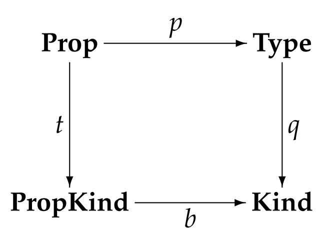
This approach is consistent with the fibration-over-fibration formulation of polymorphic logical relations in [Hera, Herb]: when formulating Reynolds's relational parametricity [Rey83] (which amounts to a *property* of the equality relation with respect to generic objects and type-quantification) only the *types* are endowed with an equality, not the *kinds*. This is evident in the syntactic formal framework of [PA93], which expresses Reynolds's relational parametricity as an additional axiom to allow formal derivation of the expected consequences, such as the existence of initial algebras and dinaturality.

In contrast, Dunphy's thesis [Dun02] develops further the reflexive-graph approach of O'Hearn and Tennent [OT93b, OT95], exploiting the cartesian closure of the 2-category of reflexive graphs of categories. Dunphy succeeds in capturing the relationally-parametric type quantifier (as formulated in [OR00, Section 7]) as a "small product" (right adjoint to a diagonal); nevertheless, in our opinion, a full-fledged categorical account of "relational parametricity" which would reconcile these various approaches is still lacking.

A Appendix: A Relational Setting for Polymorphism

We will sketch here how a fibration in $\mathcal{F}ib$ provides a categorical model of "logical relations" over a polymorphic lambda calculus.

We begin with the commuting diagram of fibrations discussed in the Introduction:

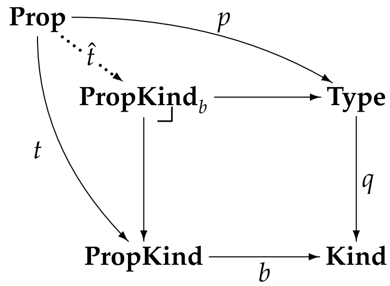


We require the following:

- q: **Type** $\longrightarrow$ **Kind** is a $\lambda 2$-fibration (i.e., a fibered cartesian-closed category with simple Ω-products and a generic object[3] T); this interprets the $\lambda 2$ term language in the usual way.
- b: **PropKind** $\longrightarrow$ **Kind** is to have fibered finite products,
- p: **Prop** $\longrightarrow$ **Type** (as a fibration over b) is a fibered cartesian-closed category with simple products, and
- t: **Prop** $\longrightarrow$ **PropKind** (as a fibration over q) has fibered simple products and a fibered generic object $\Omega_{\Gamma \vdash A}$ for every type expression A well-formed in kind context Γ; for every kind Γ, the fiber $\mathbf{PropKind}_{\Gamma}$ is a "many-sorted" Lawvere theory on such objects.

Furthermore, by factorizing through the pullback of b and q, we have

[3] The appropriate definition of a generic object is an object T in **Type** above Ω such that, for every X in **Type**, there is a cartesian morphism from X to T; in [Jac99], this is termed a "weak" generic object.



Regarding p as a fibration over b amounts to regarding $\hat{t}\colon \mathbf{Prop} \longrightarrow \mathbf{PropKind}_b$ as a fibration over **PropKind**. For a fixed proposition-kind context $\Gamma \vdash \Psi$, the fiber fibration

$$\hat{t}_{\Gamma\vdash\Psi}\colon \mathbf{Prop}_{\Gamma\vdash\Psi} \longrightarrow (\mathbf{PropKind}_b)_{\Gamma\vdash\Psi} = \mathbf{Type}_\Gamma$$

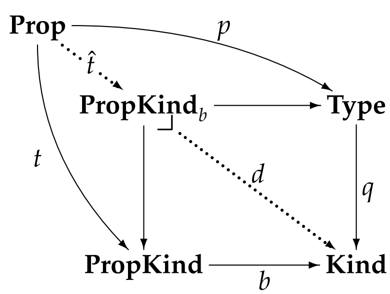
corresponds to a predicate logic over the theory of types formed in kind Γ: the fibered-ccc structure interprets the propositional logical connectives, and a first-order quantification $\forall x\colon A.\,\varphi$ over an individual variable in type context Θ is interpreted using simple products with respect to the projection from $[\![\Theta]\!] \times [\![A]\!]$ to $[\![\Theta]\!]$.

Regarding t as a fibration over q amounts to regarding $\hat{t}\colon \mathbf{Prop} \longrightarrow \mathbf{PropKind}_b$ as a fibration over **Type**. For a fixed type context $\Gamma \vdash \Theta$, the fiber fibration

$$\hat{t}_{\Gamma\vdash\Theta}\colon \mathbf{Prop}_{\Gamma\vdash\Theta} \longrightarrow (\mathbf{PropKind}_b)_{\Gamma\vdash\Theta} = \mathbf{PropKind}_\Gamma$$

has a generic object over $\Omega_{\Gamma\vdash A}$, which is the object of predicates of type $\Gamma \vdash A$ in the sense that its "elements" classify the predicates of that type. In particular, Ω_T is the object of all predicates over the "generic" type and hence classifies all predicates. A quantification $\forall R \subset B.\,\varphi$ over a predicate (or relation) variable R in proposition-kind context Ψ is interpreted by the fibered simple product in $\hat{t}$ with respect to the projection from $[\![\Psi]\!] \times \Omega_{[\![B]\!]}$ to $[\![\Psi]\!]$.

Finally, consider the diagonal fibration $d : \mathbf{PropKind}_b \longrightarrow \mathbf{Kind}$:



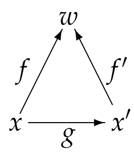
The fibration $\hat{t}\colon \mathbf{Prop} \longrightarrow \mathbf{PropKind}_b$ has cartesian simple products with respect to this diagonal. Quantification $\forall X.\,\varphi$ over a type variable X in kind context Γ is interpreted by this simple product with respect to the projection from $[\![\Gamma]\!] \times \Omega$ to $[\![\Gamma]\!]$. The limitation to cartesian simple products means that neither the type context nor the proposition-kind context can have free occurrences of the quantified type variable.

B Appendix: Proofs

B.1 Proof of Proposition 1

For any small category $\mathbf{W}$*, if* $\boldsymbol{\mathcal{S}}$ *is cartesian closed and complete, so is* $S(w)$ *for every* $w \in \mathbf{W}$.

Proof. Consider any functors $F, G \colon \mathbf{W}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}et}$; $(F \Rightarrow G)(w)$ for any w in $\mathbf{W}$ is the set of all natural transformations from $dom_w\,;F$ to $dom_w\,;G$, where $dom_w \colon (\mathbf{W}/w)^{\mathrm{op}} \longrightarrow \mathbf{W}^{\mathrm{op}}$ is the forgetful functor that maps



to $g \colon x \longrightarrow x'$. It is easily verified that this is equivalent to the usual Yoneda-derived formula.

As shown in [Nel81], this may be generalized to any complete and cartesian closed category $\boldsymbol{\mathcal{S}}$ in place of $\boldsymbol{\mathcal{S}et}$. For $F, G \colon \mathbf{W}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}}$, the natural transformations from F to G may be internalized in $\boldsymbol{\mathcal{S}}$ as the following object: $\int_w [Fw \Rightarrow Gw]$, where the $\Rightarrow$ is exponentiation in $\boldsymbol{\mathcal{S}}$ and the "integral" denotes the *end* [Mac71, Section IX.5] of the bivariant functor $(w_0, w_1) \mapsto Fw_0 \Rightarrow Gw_1$; the completeness of $\boldsymbol{\mathcal{S}}$ is needed to construct this end. Using this internalization of natural transformations, we may express $(F \Rightarrow G)(w)$ as

$$\int_{f:x \longrightarrow w} \big[(dom_w\,;F)(f) \Rightarrow (dom_w\,;G)(f)\big] \;=\; \int_{f:x \longrightarrow w} \big[Fx \Rightarrow Gx\big]$$

Products and other limits in $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ are constructed pointwise.

Finally we note that $S(w)$ is of the form $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ where $\mathbf{W}$ is itself a slice category. ■

Corollary 9 *If $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}$ are complete and cartesian closed categories and functor $F \colon \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{T}}$ preserves this structure, the post-composition functor $\boldsymbol{\mathcal{C}at}(\mathbf{W}, F) \colon \boldsymbol{\mathcal{C}at}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \boldsymbol{\mathcal{C}at}(\mathbf{W}, \boldsymbol{\mathcal{T}})$ preserves completeness and cartesian closure for any small category $\mathbf{W}$.*

B.2 Proof of Proposition 2

If $\boldsymbol{\mathcal{S}}$ is complete and cartesian closed, the fibration $p \colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \mathbf{W}$ is fibrewise cartesian closed and this structure is preserved by re-indexing.

Proof. Each fiber is cartesian closed (Proposition 1). It remains to show that, for every $h \colon w \longrightarrow w'$ in $\mathbf{W}$, the corresponding re-indexing functor preserves this structure.

Lemma 10 *If $\boldsymbol{\mathcal{S}}$ is a complete and cartesian closed category and $p \colon \mathbf{W}' \longrightarrow \mathbf{W}$ is a discrete fibration (i.e., every fiber is discrete), precomposition with p*

$$p\,;\_ \colon \boldsymbol{\mathcal{C}at}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \boldsymbol{\mathcal{C}at}(\mathbf{W}', \boldsymbol{\mathcal{S}})$$

preserves cartesian closure.

Proof of Lemma 10. Given functors $G, H \colon \mathbf{W} \longrightarrow \boldsymbol{\mathcal{S}}$, and an object w' of $\mathbf{W}'$, we must show that the canonical comparison between

$$[G \Rightarrow H](pw') = \int_{f:x \longrightarrow pw'} [Gx \Rightarrow Hx]$$

and

$$\big[(p\,;G) \Rightarrow (p\,;H)\big](w') = \int_{g:x' \longrightarrow w'} \big[(p\,;G)(x') \Rightarrow (p\,;H)(x')\big]$$

is an isomorphism. But because p is a discrete fibration, $p/w' \colon \mathbf{W}'/w' \longrightarrow \mathbf{W}/pw'$ is an isomorphism of categories, so that both ends are computed over isomorphic categories and on isomorphic diagrams. ■

To complete the proof of Proposition 2, we have, for any $h\colon w \longrightarrow w'$, the following isomorphism of functors into the slice $\mathbf{W}/w'$:

$$\begin{array}{ccc} \mathbf{W}/w & \xrightarrow{\;\sim\;} & (\mathbf{W}/w')/h \\ & {\scriptstyle \Sigma_h}\searrow \qquad \swarrow{\scriptstyle dom_h} & \\ & \mathbf{W}/w' & \end{array}$$

Because $dom_x\colon \mathbf{C}/x \longrightarrow \mathbf{C}$ is always a discrete fibration (obtained by applying the Grothendieck construction to the representable $\mathbf{C}(\_, x)\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}et}$), so is Σ_h, and Lemma 10 then yields the result that precomposition with Σ_h preserves cartesian closure. ■

B.3 Proof of Proposition 3

If $\boldsymbol{\mathcal{S}}$ is complete, the fibration $p\colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \mathbf{W}$ admits products; dually, if $\boldsymbol{\mathcal{S}}$ is co-complete, it admits co-products.

Proof. Recall that a fibration is said to admit products if re-indexing functors admit right adjoints, and these are pullback-stable (Beck-Chevalley condition). If $\boldsymbol{\mathcal{S}}$ is complete, every re-indexing functor admits a right adjoint (right Kan extension). These are pullback stable along *fibrations* [Her04, Prop. 2.4], and a commuting square $\begin{array}{ccc} p & \xrightarrow{\pi'} & w' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f'} \\ w & \xrightarrow[f]{} & x \end{array}$ is a pullback in $\mathbf{W}$ if and only if

$$\begin{array}{ccc} \mathbf{W}/p & \xrightarrow{\Sigma_{\pi'}} & \mathbf{W}/w' \\ {\scriptstyle \Sigma_\pi}\downarrow & & \downarrow{\scriptstyle \Sigma_{f'}} \\ \mathbf{W}/w & \xrightarrow[\Sigma_f]{} & \mathbf{W}/x \end{array}$$ is a pullback in $\boldsymbol{\mathcal{C}at}$. ■

B.4 Proof of Proposition 4

For any $\mathbf{W}$, the functor category $\mathbf{W}^{\mathrm{op}} \Rightarrow \boldsymbol{\mathcal{S}}$ is equivalent to the category of cartesian sections *of $p\colon \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \longrightarrow \mathbf{W}$; that is, all functors $s\colon \mathbf{W} \longrightarrow \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})$ such that $s\,;p = \mathrm{id}_{\mathbf{W}}$ and, for every $\mathbf{W}$-morphism f, $s(f)$ is cartesian.*

Proof. Given $F\colon \mathbf{W}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}}$, a cartesian section $s_F\colon \mathbf{W} \longrightarrow \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})$ of p may be defined by

- $s_F(w)(f\colon x \longrightarrow w) = F(x)$
- $s_F(w)\left(\begin{array}{ccc} & w & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle f'} \\ x & \xrightarrow[g]{} & x' \end{array}\right) = F(g)\colon F(x') \longrightarrow F(x)$

In the other direction, given a cartesian section $s\colon \mathbf{W} \longrightarrow \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})$ of p, the object part of a functor $F_s\colon \mathbf{W}^{\mathrm{op}} \longrightarrow \boldsymbol{\mathcal{S}}$ may be defined by $F_s(w) = s(w)(\mathrm{id}_w)$. For the morphism part, consider any $\mathbf{W}$-morphism $f\colon w \longrightarrow w'$; then $s(w)(\mathrm{id}_w) \cong \left(\Sigma_f^{\mathrm{op}}\,; s(w')\right)(\mathrm{id}_w) = s(w')(f)$, and we may define

$$F_s(f) = s(w')\left(\begin{array}{ccc} & w' & \\ {}^{f}\nearrow & & \nwarrow^{\mathrm{id}_{w'}} \\ w & \xrightarrow[f]{} & w' \end{array}\right) : s(w')(\mathrm{id}_{w'}) \longrightarrow s(w')(f) \cong s(w)(\mathrm{id}_w)$$

■

B.5 Proof of Proposition 5

In the diagram

$$\begin{array}{ccc} \mathbf{Slices}(rw, rs) & \xrightarrow{\widetilde{rw}} & \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \times \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \times p} \\ \mathbf{RW} & \xrightarrow[rw]{} & \mathbf{W} \times \mathbf{W} \end{array}$$

$(q, p \times p)$: $\widetilde{rw} \longrightarrow rw$ is a fibration in $\boldsymbol{\mathcal{F}ib}$ whenever p and rw are fibrations.

Proof. When rw is a fibration, it suffices (see Appendix C) to show that, for each W in $\mathbf{RW}$ with $rw(W) = (w_0, w_1)$, the fiber functor $\widetilde{rw}_W$: $\mathbf{Slices}(rw, rs)_W \longrightarrow \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})_{w_0} \times \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}})_{w_1}$, taking objects $(\widetilde{F}, F_0, F_1)$ (in the fiber above W) to (F_0, F_1), is a fibration, and that such fibrations are preserved by the re-indexing functors induced by morphisms in $\mathbf{RW}$.

To show that $\widetilde{rw}_W$ is a fibration, consider any object $(\widetilde{G}, G_0, G_1)$ of $\mathbf{Slices}(rw, rs)_W$:

$$\begin{array}{ccc} (\mathbf{RW}/W)^{\mathrm{op}} & \xrightarrow{\widetilde{G}} & \boldsymbol{\mathcal{RS}} \\ {\scriptstyle (rw/W)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle rs} \\ (\mathbf{W}/w_0)^{\mathrm{op}} \times (\mathbf{W}/w_1)^{\mathrm{op}} & \xrightarrow[G_0 \times G_1]{} & \boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}} \end{array}$$

and any morphism $(\eta_0 : F_0 \dot{\rightarrow} G_0, \eta_1 : F_1 \dot{\rightarrow} G_1)$ into the underlying pair $(G_0, G_1) = \widetilde{rw}(\widetilde{G}, G_0, G_1)$:

$$\begin{array}{ccc} (\mathbf{RW}/W)^{\mathrm{op}} & \xrightarrow{\widetilde{G}} & \boldsymbol{\mathcal{RS}} \\ {\scriptstyle (rw/W)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle rs} \\ (\mathbf{W}/w_0)^{\mathrm{op}} \times (\mathbf{W}/w_1)^{\mathrm{op}} & \begin{array}{c} \xrightarrow{F_0 \times F_1} \\ \Downarrow \eta_0 \times \eta_1 \\ \xrightarrow[G_0 \times G_1]{} \end{array} & \boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}} \end{array}$$

We will construct the $\widetilde{\eta}$ component (with co-domain $\widetilde{G}$) of a cartesian lifting $(\widetilde{\eta}, \eta_0, \eta_1)$ of (η_0, η_1) *pointwise*, using the fact that rs is a fibration. Consider any object $f : X \longrightarrow W$ of $\mathbf{RW}/W$ with $rw(f) = (f_0 : x_0 \longrightarrow w_0, f_1 : x_1 \longrightarrow w_1)$; then $\widetilde{\eta}(f)$ is defined as a cartesian lifting of $\eta_0(f_0) \times \eta_1(f_1)$ with respect to fibration rs. The domain of $\widetilde{\eta}$ is a contravariant functor from $(\mathbf{RW}/W)$ to $\boldsymbol{\mathcal{RS}}$ whose action on objects yields the relevant domain of the cartesian lifting and whose actions on morphisms is determined by the universality property of the cartesian lifting.

The pointwise nature of the liftings makes them stable under the action of re-indexing functors induced by morphisms h in $\mathbf{RW}$, since the action is given by precomposition with functors Σ_h between slices. ■

B.6 Proof of Corollary 6

If rs is a fibration, so is $rw^{\mathrm{op}} \Rightarrow rs$ *for* any *functor rw.*

Proof. $rw^{\mathrm{op}} \Rightarrow rs$ is essentially $\widetilde{rw}_W$ for $W = 1$ and the pointwise construction depends only on rs being a fibration. ■

B.7 Proof of Proposition 7

If rs is a fibration of complete cartesian closed categories, the diagram

$$\begin{array}{ccc} \mathbf{Slices}(rw, rs) & \xrightarrow{\widetilde{rw}} & \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \times \mathbf{Slices}(\mathbf{W}, \boldsymbol{\mathcal{S}}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \times p} \\ \mathbf{RW} & \xrightarrow[rw]{} & \mathbf{W} \times \mathbf{W} \end{array}$$

is a morphism of fibered cartesian closed categories with products.

Proof. By Corollary 9 in Section B.1, the functor $\widetilde{rw}_W$ preserves cartesian closed structure and completeness. ■

C Appendix: Fibrations Over Fibrations

Recall that a morphism $p: E \longrightarrow B$ in any 2-category $\mathcal{K}$ is said to be a *fibration in* $\mathcal{K}$ if, for every object X, the functor $\mathcal{K}(X, p): \mathcal{K}(X, E) \longrightarrow \mathcal{K}(X, B)$ is a fibration of categories. Here we are concerned with fibrations in the 2-categories $\boldsymbol{\mathcal{F}ib}/\mathbf{B}$ of fibrations over a fixed base category $\mathbf{B}$, and $\boldsymbol{\mathcal{F}ib}$, fibrations over arbitrary base categories.

C.1 Fibrations in $\boldsymbol{\mathcal{F}ib}/\mathbf{B}$

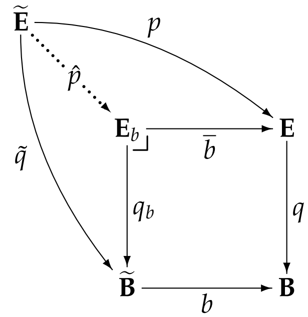
The Grothendieck correspondence between fibrations $p: \mathbf{E} \longrightarrow \mathbf{B}$ and pseudo-functors ("indexed categories") from $\mathbf{B}^{\mathrm{op}}$ to $\boldsymbol{\mathcal{C}at}$ allows us to view fibrations in $\boldsymbol{\mathcal{F}ib}/\mathbf{B}$ in indexed terms: a morphism $f: p \longrightarrow q$ between two fibrations $p: \mathbf{E} \longrightarrow \mathbf{B}$ and $q: \mathbf{D} \longrightarrow \mathbf{B}$ (with common base $\mathbf{B}$) is a fibration in $\boldsymbol{\mathcal{F}ib}/\mathbf{B}$ iff, for every object I of $\mathbf{B}$, the fiber functor $f_I: \mathbf{E}_I \longrightarrow \mathbf{D}_I$ is a fibration (of categories) and the re-indexing (substitution) functors preserve cartesian liftings between such fibrations.

C.2 Fibrations in $\boldsymbol{\mathcal{F}ib}$

Consider a morphism $(p, b): \tilde{q} \longrightarrow q$ in $\boldsymbol{\mathcal{F}ib}$:

$$\begin{array}{ccc} \widetilde{\mathbf{E}} & \xrightarrow{p} & \mathbf{E} \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ \widetilde{\mathbf{B}} & \xrightarrow[b]{} & \mathbf{B} \end{array}$$

This may be factorized through the pullback of b and q:



Hermida [Hera] proves that $(p, b)\colon \tilde{q} \longrightarrow q$ is a fibration in $\boldsymbol{\mathcal{F}ib}$ iff $\hat{p}\colon \tilde{q} \longrightarrow q_b$ is a fibration in $\boldsymbol{\mathcal{F}ib}/\widetilde{\mathbf{B}}$. Thus, in elementary terms, to give a fibration in $\boldsymbol{\mathcal{F}ib}$, we must provide a collection of fibrations $p_I\colon \widetilde{\mathbf{E}}_I \longrightarrow \mathbf{E}_{b(I)}$ for all $I \in \widetilde{\mathbf{B}}$ such that the re-indexing functors for $\tilde{q}$ and p preserve the cartesian liftings of all such p_I.

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