Descent on 2-fibrations and strongly 2-regular 2-categories

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Abstract. We consider pseudo-descent in the context of 2-fibrations. A 2-category of descent data is associated to a 3-truncated simplicial object in the base 2-category. A morphism q in the base induces (via *comma-objects* and pullbacks) an internal category whose truncated simplicial nerve induces in turn the 2-category of descent data for q. When the 2-fibration admits direct images, we provide the analogous of the Beck-Bénabou-Roubaud theorem, identifying the 2-category of descent data with that of pseudo-algebras for the pseudo-monad $q^*\Sigma_q$. We introduce a notion of *strong 2-regularity* for a 2-category $\mathcal{R}$, so that its basic 2-fibration of internal fibrations $cod : \mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$ admits direct images. In this context, we show that *essentially-surjective-on-objects* morphisms, defined by a certain lax colimit, are of effective descent by means of a Beck-style pseudo-monadicity theorem.

Keywords: 2-fibration, effective descent, essentially-surjective-on-objects, pseudo-monadicity

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1. Introduction

The theory of descent concerns itself with the (re)construction of global data from local information. The simplest instance is the theory of *sheaves*, which (re)constructs a family of sets over a topological space (for example, the family of continuous real-valued functions) from families over a covering of such space, provided such families agree (are coherently isomorphic) on overlaps. When the data is a family of categories rather than sets, we talk about *stacks*. The general formulation of descent involves the framework of *fibrations* as an abstract axiomatics of 'family': for a category $\mathbb{E}$ fibred over a base category $\mathbb{B}$, the fibre $\mathbb{E}_I$ over an object I of the base is the category of 'abstract I-families'.

The 'local information' referred to above is called a *descent datum*. For a fibration $p : \mathbb{E} \to \mathbb{B}$, with associated pseudo-functor $\mathcal{F}_p : \mathbb{B}^{op} \to \mathcal{C}at$, it is formulated relative to a 2-truncated simplicial object in $\mathbb{B}$ induced by a morphism $q : C \to I$ as follows (*cf.*(Giraud, 1964; Janelidze and Tholen, 1997)): the kernel of q, $\underline{\mathsf{Cat}}_q$ (the pullback of q along itself), is an equivalence relation and consequently, an internal groupoid in $\mathbb{B}$. Applying $\mathcal{F}_p$ to the 2-truncation of the simplicial nerve of $\underline{\mathsf{Cat}}_q$ yields a pseudo-diagram in $\mathcal{C}at$ from which we obtain a corresponding category of descent data $\mathsf{Des}(\underline{\mathsf{Cat}}_q)$ as a kind of weighted bilimit. The morphism q is said to be of effective descent when the change-of-base functor $q^* : \mathbb{E}_I \to \mathbb{E}_C$ induces an equivalence between $\mathbb{E}_I$ and $\mathsf{Des}(\underline{\mathsf{Cat}}_q)$. In this situation, data in the fibre $\mathbb{E}_I$ can be (re)constructed from descent data.

In order to tackle the same problem for 2-dimensional data, we work in the framework of *2-fibrations*. A 2-fibration is a 2-functor $P : \mathcal{E} \to \mathcal{B}$ with a (contravariant) lifting property for both 1-cells and 2-cells, as introduced in (Hermida, 1999) (see §2).

We focus here on two basic results of 1-dimensional descent theory (see (Janelidze and Tholen, 1994) for a good account of this part of the theory) which we (re)prove at the 2-dimensional level:

- When the fibration p admits *direct images*, a classical result of Bénabou and Roubaud (independently credited to Beck) identifies the category of descent data $\mathsf{Des}(\underline{\mathsf{Cat}}_q)$ with the category of algebras for the monad $q^*\Sigma_q : \mathbb{E}_C \to \mathbb{E}_C$.
- When $\mathbb{B}$ is a *regular* category (one admitting pullbacks and pullback stable coequalisers-of-kernel-pairs), the fibration of subobjects $cod : Sub(\mathbb{B}) \to \mathbb{B}$ admits direct images satisfying Beck-Chevalley (for a category with pullbacks, this latter fact is one of the various equivalent formulations of regularity). For this fibration, regular epimorphisms are of effective descent. If we have additionally coequalisers in the slice categories $\mathbb{B}/X$, preserved by pullbacks along

regular epis, regular epis are of effective descent for the basic fibration $cod : \mathbb{B}^{\rightarrow} \rightarrow \mathbb{B}$ as well. In both cases, the result is proved via the above reduction of effective descent to monadicity. The pullback stability of coequalisers-of-kernel-pairs $q : E \rightarrow A$ ensure that the change-of-base functor $q^* : \mathbb{B}/A \rightarrow \mathbb{B}/E$ reflects both strong epis and monos, and is therefore conservative. The existence of the relevant split coequalisers follows directly from the hypothesis.

In order to develop the 2-dimensional theory, we need the following ingredients:

– A notion of direct images for 2-fibrations (§2.1). Of course, direct images are left biadjoints to change-of-base. The only subtlety is finding the appropriate notion of stability for these adjoints, which in the 1-dimensional case is the well-known Beck-Chevalley condition over pullback squares. Here the condition is formulated over comma-objects (following the analysis of the computation of pointwise left Kan extensions). It entails stabilty of direct images for pullbacks along *cofibrations* (Proposition 2.4).

– The construction of the relevant 3-truncated simplicial object associated to a morphism §3.2. Here we have followed the approach which authors working on 2-dimensional descent for toposes have adopted (*cf.*(Moerdijk and Vermeulen, 2000; Makkai, 1993)): associate to a morphism $p : E \rightarrow B$ an internal category given by its '2-kernel', *i.e.* the comma-object $p \downarrow p$ with its induced internal-category structure. We motivate our formulation of 2-dimensional descent via an analysis of fibrations and functors (basic examples of 2-dimensional data which we may analyse via descent) as lax colimits in §3 (see Proposition 3.1).

– The identification of the 2-category $\mathsf{Des}(\underline{\mathsf{Cat}}_q)$ of 'pseudo' descent data associated to p with the 2-category of pseudo-algebras for the pseudo-monad $q^*\Sigma_q : \mathcal{E}_T \rightarrow \mathcal{E}_T$ (Theorem 4.2).

– The relevant axioms of strong 2-regularity (§5) in a 2-category $\mathcal{R}$ so as to have a basic 2-fibration of internal fibrations (the 2-fibration of families or '$\mathcal{R}$ 2-fibred over itself'), $cod : \mathcal{F}ib(\mathcal{R}) \rightarrow \mathcal{R}$ with direct images (Proposition 5.9).

– A suitable notion of 'covering' morphism in this strongly 2-regular context. In analogy with the 1-dimensional situation, where we consider the kernel of a morphism and its quotient (regular epi), here we consider the 'lax colimit of $p \downarrow p$' (Definition 5.1) and show that the so-called essentially-surjective-on-objects (eso) morphisms are of effective descent (Theorem 5.13).

There are different notions of regularity for 2-categories in the literature, notably in (Street, 1982; Carboni et al., 1994), which have different goals than ours. We were motivated by descent: in a category with pullback stable coequalisers, regular epis are effective descent morphisms for the basic fibration. Here we achieve this result for the basic 2-fibration considering the esos as our 'regular epis'. Although we do not claim that our axiomatisation is a definitive one (for instance, we have not dealt with the attendant factorisation systems), the developments here do complement the results on 2-dimensional regularity in *ibid.*

We also take the opportunity to present a pseudo-monadicity theorem (à la Beck), Theorem A.2. The main point here is the identification of the right kind of 'pseudo-split bicolimit' to deal with pseudo-algebras. This is an auxiliary piece of machinery which we use to establish the effectiveness of an eso, as in classical descent theory for exact categories. Being of interest on its own, we have relegated it to an appendix.

2. 2-fibrations

We recall from (Hermida, 1999) the definition of a 2-fibration. We choose the simplest of the various equivalent characterisations in Theorem 2.8 of *ibid.*

2.1. Definition. Let $P : \mathcal{E} \to \mathcal{B}$ be a 2-functor.

1. A 1-cell $f : X \to Y$ in $\mathcal{E}$ is *1-cartesian* if it is cartesian in the usual sense for the underlying functor $P_0 : \mathcal{E}_0 \to \mathcal{B}_0$, *i.e.* for any 1-cell $h : Z \to Y$ with $Ph = Pf \circ u$ for some given 1-cell $u : PZ \to PX$, there is a unique 1-cell $\hat{h} : Z \to X$ with $P\hat{h} = u$ and $f \circ \hat{h} = h$.

2. A 1-cell $f : X \to Y$ in $\mathcal{E}$ is *2-cartesian* if it is 1-cartesian and for any 2-cell $\alpha : g \Rightarrow h : Z \to Y$ such that

$$PZ \underset{Ph}{\overset{Pg}{\Downarrow P\alpha}} PY \quad = \quad PZ \underset{v}{\overset{u}{\Downarrow \sigma}} PX \xrightarrow{Pf} PY$$

   there is a unique 2-cell $\phi : \hat{g} \Rightarrow \hat{h} : Z \to X$ such that $f\phi = \alpha$ and $P\phi = \sigma$. Here, the 1-cells $\hat{g}$ and $\hat{h}$ are uniquely determined because f is 1-cartesian.

3. A 2-functor $P : \mathcal{E} \to \mathcal{B}$ is a **2-fibration** if it satisfies the following:

- for any object X in $\mathcal{E}$ and any 1-cell $u : I \to PX$ in $\mathcal{B}$, there is a 2-cartesian 1-cell $\overline{u} : u^*(X) \to X$ with $P\overline{u} = u$.
- For every pair of objects X, Y in $\mathcal{E}$, the corresponding functor between the hom-categories, $P_{X,Y} : \mathcal{E}(X,Y) \to \mathcal{B}(PX, PY)$, is a fibration (of categories, in the usual sense). Furthermore, for every 1-cell $h : Z \to X$ in $\mathcal{E}$, the precomposition functor $\mathcal{E}(h,Y) : \mathcal{E}(X,Y) \to \mathcal{E}(Z,Y)$ preserves cartesian morphisms (from $P_{X,Y}$ to $P_{Z,Y}$).

The stability of the (local) cartesian 2-cells under precomposition with 1-cells means that cartesian 2-cells have a componentwise nature. As expected, given a 2-fibration $P : \mathcal{E} \to \mathcal{B}$, a choice of cartesian liftings of 1-cells yields a *substitution*, *change-of-base* or *reindexing* 2-functor between (2-)fibres associated to each 1-cell

$$(u : I \to J) \longmapsto (u^* : \mathcal{E}_J \to \mathcal{E}_I)$$

and likewise, a choice of cartesian 2-cells (in the hom-fibrations) yields a *pseudo-natural* transformation for every 2-cell

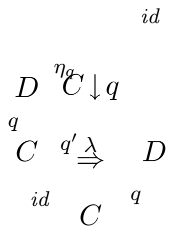
$$(\sigma : u \Rightarrow v : I \to J) \longmapsto (\sigma^* : v^* \Rightarrow u^* : \mathcal{E}_J \to \mathcal{E}_I)$$

and these data assemble into a *homomorphism of tricategories*[1] $\mathcal{F}_P : \mathcal{B}^{coop} \to 2\text{-}\mathcal{C}at_\otimes$, where $2\text{-}\mathcal{C}at_\otimes$ is the *Gray-category* of 2-categories, 2-functors, pseudo-natural transformations and modifications, *cf.* (Gordon et al., 1995; Day and Street, 1997; Lack, 2000).

We advise the reader to keep in mind as concrete example the 2-fibration $cod : \mathcal{F}ib \to \mathcal{C}at$, whose fibre over a category $\mathbb{I}$ is the 2-category of fibrations with base $\mathbb{I}$; the cartesian 1-cells are obtained by pullbacks, and the cartesian 2-cells are constructed componentwise using cartesian liftings in the target fibration. See (Hermida, 1999) for details.

We will be interested in the 2-fibration $cod : \mathcal{F}ib(\mathcal{K}) \to \mathcal{K}$ of fibrations in a 2-category $\mathcal{K}$. Recall from (Street, 1973), that the notion of fibration in a 2-category is *representable*, *i.e.* a morphism $q : D \to C$ is a fibration iff for every object X, the functor $\mathcal{K}(X,q) : \mathcal{K}(X,D) \to \mathcal{K}(X,C)$ is a fibration of categories. Furthermore, when $\mathcal{K}$ admits comma-objects, the notion can be internalised, so that the property of q being a fibration amounts to the following: given the diagram

[1] We should point out that the coherent 2-cells for associativity and units are pseudo-natural *isomorphisms* rather than equivalences in this case.



where the the 2-cell λ corresponds to a comma-object, the morphism q is a fibration iff $\eta_q : q \to q'$ has a right-adjoint in $\mathcal{K}/C$. If $\mathcal{K}$ admits pullbacks along fibrations, the 2-functor $cod : \mathcal{F}ib(\mathcal{K}) \to \mathcal{K}$, which takes a fibration to its base, is a 2-fibration just like our original example over $\mathcal{C}at$.

2-fibrations of groupoids

A referee has brought to our attention the references (Moerdijk, 1990; Hardie et al., 2002), which introduce fibrations for 2-groupoids and bigroupoids respectively. For the benefit of homotopically minded readers who have not come across 2-fibrations before, we point out that our notion of 2-fibration reduces to the notions in *ibid.* when the total and base 2-categories are groupoidal.

We recall that (Moerdijk, 1990) considers a *2-groupoid* as a 2-category in which every morphism and 2-cell is an iso. Grothendieck fibrations of 2-groupoids are characterised in Lemma 3 of *ibid.* for strict homomorphisms (2-functors) as those $\rho : \mathcal{B} \to \mathcal{A}$ which are fibrations at dimension 1 and the groupoid morphisms $\rho_{B,B'} : \mathcal{B}(B, B') \to \mathcal{A}(\rho B, \rho B')$ are fibrations for every pair of objects B, B'. It is evident thus that this notion is exactly the instantiation of 2-fibration to the 2-groupoidal case.

For a pseudo-functor $p : \mathcal{E} \to \mathcal{B}$, we use coherence (the bireflection of 2-categories and pseudo-functors into 2-categories and 2-functors) to factorise it as $p = \overline{p}\eta_{\mathcal{E}}$, where $\overline{p} : \overline{\mathcal{E}} \to \mathcal{B}$ is a 2-functor and $\eta_{\mathcal{E}} : \mathcal{E} \to \overline{\mathcal{E}}$ is a biequivalence surjective on objects and morphisms (and is therefore a fibration). Now we may define p to be a 2-fibration if $\overline{p}$ is such. This would entail a lifting of 1-cells 'up to 2-isomorphism', but with the lifting property of 2-cells, such lifting can be made strictly surjective, as we have considered before. Therefore, when instantiated to the 2-groupoidal case, our notion agrees with that of (Moerdijk, 1990, Def. 1.7).

As for (Hardie et al., 2002), their definition of *bigroupoid* seems to be a bicategory whose 1-cells are equivalences and its 2-cells are isos. Since we have not defined 2-fibrations for bicategories, we have to extend our definition to bicategories for a precise comparison. Just like we did above for pseudo-functors, the most sensible approach to this notion would be via coherence: a homomorphism of bicategories $F : \mathcal{E} \to \mathcal{B}$ must be a 2-fibration if its associated 2-functor between 2-categories $\overline{\eta_{\mathcal{A}} F} : \overline{\mathcal{E}} \to \overline{\mathcal{B}}$ is such. This is consistent with Proposition 2.3 of *ibid.*, according to which a fibration of bigroupoids induces a fibration of their associated Poincaré groupoids. Notice that this transformation entails once again a lifting of 1-cells 'up to 2-isomorphism', which combined with the lifting of 2-cells, yields in turn a strictly surjective lifting of 1-cells. A fortriori, our given definition of 2-fibration seems adequate as it stands for homomorphisms of bicategories. We conclude therefore that 2-fibrations of bicategories instantiated in the bigroupoidal case would agree with those in Def. 2.1 of *ibid.*

2.1. The Beck-Chevalley condition for direct images

For an ordinary fibration $p : \mathbb{E} \to \mathbb{B}$, *direct images* or *sums* amount to the existence of cocartesian liftings, with the additional requirement that such liftings be stable under pullback along cartesian morphisms. Making a choice of cartesian liftings for p, these properties correspond to the existence of left adjoints to substitution functors, $\Sigma_u \dashv u^*$, satisfying the Beck-Chevalley condition (the left adjoints are stable under pullback). Given our intended application of direct images in descent problems, we would now give the corresponding formulation for 2-fibrations in the latter fashion.

In order to determine the relevant stability condition in the 2-dimensional context, we analyse our basic example $cod : \mathcal{Fib} \to \mathcal{Cat}$. Given a functor $u : \mathbb{I} \to \mathbb{J}$ and a fibration $x : \mathbb{X} \to \mathbb{I}$, we regard it as a pseudo-functor $\mathcal{F}_x : \mathbb{X}^{op} \to \mathcal{Cat}$ and compute its left Kan extension along u; its comprehended version (Grothendieck construction $\int$) yields a direct image:

$$\Sigma_u(x) = \int Lan_u(\mathcal{F}_x)$$

This Kan extension is computed *pointwise* by means of colimits and *comma-objects*: for an object $j \in \mathbb{J}$

$$\Sigma_u(x)_j = \varinjlim((j \downarrow u)^{op} \to \mathbb{I}^{op} \xrightarrow{\mathcal{F}_x} \mathcal{Cat})$$

where the following diagram

$$\begin{array}{ccc} & u^{op} \downarrow j_q & \\ \mathbb{I}^{op} & \overset{\lambda}{\Rightarrow} & \mathbf{1} \\ {\scriptstyle (u)^{op}} & \mathbb{J}^{op} & {\scriptstyle j} \end{array}$$

is a comma-object. Eliminating ^{op}, we get the following comma-object

$$\begin{array}{ccc} & j \downarrow u \;{\scriptstyle q} & \\ \mathbf{1} & \overset{\lambda}{\Rightarrow} & \mathbb{I} \\ {\scriptstyle j} & \mathbb{J} & {\scriptstyle u} \end{array}$$

2.2. Definition. A 2-fibration $P : \mathcal{E} \to \mathcal{B}$ admits **direct images** if, for every 1-cell $u : I \to J$ in $\mathcal{B}$, the induced 2-functor $u^* : \mathcal{E}_J \to \mathcal{E}_I$ admits a left bi-adjoint, that is a pseudo-functor $\Sigma_u : \mathcal{E}_I \to \mathcal{E}_J$ together with pseudo-natural transformations $\eta : id \Rightarrow u^*\Sigma_u$ and $\epsilon : \Sigma_u u^* \Rightarrow id$ satisfying the usual triangular identities up to coherent invertible modifications *cf.*(Lack, 2000; Marmolejo, 1999). Equivalently, for the given objects $\Sigma_u X$, there are equivalences of categories

$$\mathcal{E}_J(\Sigma_u X, Y) \simeq \mathcal{E}_I(X, u^*Y)$$

pseudo-natural in X and Y.

Furthermore, we require that such left biadjoints be stable under comma-objects: given a comma-object

$$\begin{array}{ccccc} & {}_{p} & v\downarrow u & {}_{q} & \\ K & & \overset{\lambda}{\Rightarrow} & & I \\ & {}^{v} & J & {}^{u} & \end{array}$$

in $\mathcal{B}$, the canonical comparison pseudo-natural transformation $\hat{\lambda} : \Sigma_p q^* \Rightarrow v^*\Sigma_u$ (the transpose of $\lambda^* : q^*u^* \Rightarrow p^*v^*$) must be an equivalence.

2.3. Remark. The above Beck-Chevalley-condition-on-comma-objects occurs also in Grothendieck's *derivateurs* (pseudo-functors $F : \mathbb{D} \to \mathcal{C}at$ where $\mathbb{D}$ is a sub-2-category of $\mathcal{C}at$ closed under certain limits), but only for global elements, in line with the analysis preceding our definition. Since we are working over a general base 2-category rather than a sub-2-category of $\mathcal{C}at$, we are led to impose the condition on arbitrary comma-objects.

We next show that the above condition for comma-objects implies the usual Beck-Chevalley condition for direct images when pulling back along a cofibration:

2.4. Proposition. *Consider the following pullback square in $\mathcal{B}$*

$$\begin{array}{ccc} Q & \overset{\tilde{d}}{} & K \\ {}^{\tilde{c}} & & {}^{c} \\ I & {}_{d} & J \end{array}$$

where c is a cofibration ('covariant fibration'). If the 2-fibration $P : \mathcal{E} \to \mathcal{B}$ has direct images, the canonical comparison pseudo-natural transformation

$$\kappa : \Sigma_{d'}(c')^* \Rightarrow c^*\Sigma_d$$

is an equivalence.

Proof. Consider the diagram

$$\begin{array}{ccccc} Q & \tilde{d} & K & & \\ \tilde{c} & \tilde{\eta}_c & \eta_c & c & id \\ c\downarrow d & & c\downarrow J & & K \\ c' & d' & c' & \Leftarrow & c \\ I & d & J & id & J \end{array}$$

where both squares on the left are pullbacks and the right and front squares are comma-objects. Since c is a cofibration, there is a left adjoint $\kappa_c \dashv \eta_c$ with unit $\rho : id \Rightarrow \eta_c\kappa_c$ satisfying $c'\rho = id_{c'}$ (therefore $\kappa_c^*\eta_c^*(c')^* \overset{(c'\rho)^*}{\simeq} (c')^*$). This adjunction implies that $\kappa_c^* : \mathcal{E}_K \to \mathcal{E}_{C\downarrow J}$ is left biadjoint to η_c^*, so that $\kappa_c^* \simeq \Sigma_{\eta_c}$. By the same argument, $\tilde{\eta}_c$ has a left adjoint $\tilde{\kappa}_c$ with the same properties (namely, the pullback $d^*\kappa_c$ of κ_c along d). We have the following chain of equivalences

$$\begin{aligned} \Sigma_{\tilde{d}}(\tilde{c})^* &\simeq \Sigma_{d'}\Sigma_{\tilde{\eta}_c}(\tilde{\eta}_c)^*(c'')^* \\ &\simeq \Sigma_{d'}(\kappa_c)^*(\tilde{\eta}_c)^*(c'')^* \\ &\simeq \Sigma_{d'}(c'')^* \\ &\simeq c^*\Sigma_d \end{aligned}$$

where the last equivalence is the comma-object condition on direct images relative to the front comma-object. □

3. Pseudo-Descent

We consider a 2-fibration $P : \mathcal{E} \to \mathcal{B}$ in which the base $\mathcal{B}$ admits comma objects and pullbacks. For simplicity we assume, wlog, that our 2-fibration is *normalised*, *i.e.* $id^* = id$ for both morphisms and 2-cells. The *pseudo-descent* problem amounts to (re)constructing data in a fibre 2-category $\mathcal{E}_X$ based on suitably structured data (*descent data*) on a 'covering' of the object X.

In the case of ordinary fibrations, mainly for applications in topos theory, it is usual to consider a covering as given by a regular epimorphism $q : \tilde{X} \to X$, and the relevant descent data is determined by its kernel groupoid, which yields a 2-truncated simplicial object in the base. A motivating example is that where the base category is that of topological spaces, $\mathcal{T}op$. A covering $\{U_i\}_{i\in I}$ of a space X gives rise to an epimorphism $q : \tilde{X} = \coprod_{i\in I} U_i \to X$. A family $\mathcal{F}$ over $\tilde{X}$ consists of a collection of families $\mathcal{F}_i$'s over the U_i's. The restrictions $d^*\mathcal{F}$ and $c^*\mathcal{F}$ of $\mathcal{F}$ along the two projections out of the pullback of q with itself, $\mathsf{Ker}_q \overset{d}{\underset{c}{\rightrightarrows}} \tilde{X}$, are the collections of restrictions of the $\mathcal{F}_i$'s to the intersections (or overlaps) $U_i \cap U_j$: if $\mathcal{F}_{ij}$ denotes the restriction of $\mathcal{F}_i$ to such intersection, we must have isomorphisms $\theta_{ij} : \mathcal{F}_{ij} \xrightarrow{\sim} \mathcal{F}_{ji}$ between both restrictions to common intersections. Furthermore, such isomorphisms should satisfy the so-called cocycle conditions, which express consistency when restricting to common triple intersections. Such a family of coherent isomorphisms becomes a single isomorphism of families $\theta : d^*\mathcal{F} \xrightarrow{\sim} c^*\mathcal{F}$, and the cocycle conditions can be expressed by a suitable equation among the restrictions of θ to the third level of the simplicial object corresponding to the kernel groupoid of q. This reformulation applies to abstract families and we can thus make sense of descent relative to an arbitrary fibration over a category $\mathbb{B}$ which admits the construction of the internal kernel-groupoid associated to a morphism, *e.g.* when $\mathbb{B}$ admits pullbacks.

Since we deal with one additional dimension, (pseudo-)descent data would be determined by a 3-truncated simplicial object (see §3.1). By means of comma-objects and pullbacks, a morphism in $\mathcal{B}$ determines an internal category. Its nerve gives the relevant (3-truncated) simplicial object to consider pseudo-descent data for such a morphism and define the condition of *effective descent* relative to the 2-fibration P (§3.2).

In order to motivate such formulation, let us revisit (and later generalise) the structural decomposition of a *fibred category as a lax colimit of its fibres*. This is a typical kind of 2-dimensional descent situation we want to consider: descent for the 2-fibration $cod : \mathcal{F}ib \to \mathcal{C}at$ of fibrations varying over their base category.

Consider a fibration $p : \mathbb{E} \to \mathbb{B}$. Choosing a cleavage, a morphism $f : I \to J$ in the base induces a reindexing (or change-of-base) functor $f^* : \mathbb{E}_J \to \mathbb{E}_I$ between the fibres and a 2-cell

$$\begin{array}{ccc} \mathbb{E}_J & \xrightarrow{f^*} & \mathbb{E}_I \\ & \theta_f & \\ {\scriptstyle \iota_J} \searrow & & \swarrow {\scriptstyle \iota_I} \\ & \mathbb{E} & \end{array}$$

where $\iota_I : \mathbb{E}_I \to \mathbb{E}$ is the inclusion of the fibre in the total category and the 2-cell θ_f has component at an object $x \in \mathbb{E}_J$ the (chosen) cartesian lifting of f at x, $\theta_f x : f^*x \to x$. The pseudo-functorial nature of the reindexing functors means that we get, for composable morphisms f and g, isomorphisms $\delta_{f,g} : g^*f^* \Rightarrow (gf)^*$ satisfying the evident coherent conditions. We thus get a pseudo-functor $\mathcal{F}_p : \mathbb{B}^{op} \to \mathcal{C}at$ associated to p, and the above diagram of ι's and θ's exhibit $\mathbb{E}$ as a **lax colimit** of $\mathcal{F}_p$ (Grothendieck, 1971). Furthermore, we recover the functor $p : \mathbb{E} \to \mathbb{B}$ as uniquely induced by the (trivial) lax cocone into $\mathbb{B}$ taking anything in a fibre to its base object (the functor p itself is a lax colimit in the 2-category $\mathcal{C}at/\mathbb{B}$).

A slightly more involved variant applies to arbitrary functors into a given category: there is an equivalence of 2-categories

$$\mathcal{C}at/\mathbb{B} \simeq \mathsf{Lax}_{rep}[\mathbb{B}, \mathsf{Bimod}(\mathcal{C}at)]$$

which generalises the one above between fibrations and pseudo-functors. The right-hand side is the bicategory whose objects are normal lax functors into the bicategory of categories and bimodules (two-sided discrete fibrations) and whose morphisms are lax transformations with *representable bimodules* (induced by functors) as components (see (Hermida, 2001, §11) and (Hermida, 2002) for details). The correspondence from left to right associates to a functor $a : \mathbb{A} \to \mathbb{B}$ the lax functor $\Lambda_a : \mathbb{B} \to \mathsf{Bimod}(\mathcal{C}at)$ which assigns to an object I of $\mathbb{B}$ the fibre $\mathbb{A}_I$, and to a morphism $f : I \to J$, the bimodule $\mathbb{A}_f : \mathbb{A}_I \not\to \mathbb{A}_J$ with values

$$\mathbb{A}_f(x, y) = \{h : x \to y \in \mathbb{A} \mid ah = f\}$$

Just as before, we have a 2-cell

$$\begin{array}{ccc} & \mathbb{A}_f & \\ \mathbb{A}_I & & \mathbb{A}_J \\ & \theta_f & \\ (\iota_I)_\# & & (\iota_J)_\# \\ & \mathbb{A} & \end{array}$$

in $\mathsf{Bimod}(\mathcal{C}at)$, where $u_\# : \mathbb{X} \to \mathbb{Y}$ denotes the representable bimodule induced by a functor $u : \mathbb{X} \to \mathbb{Y}$. The 2-cell θ_f is given as follows: given objects $x \in \mathbb{A}_I$ and $z \in \mathbb{A}$, its component $\theta_f(x, z)$ maps the equivalence class of a pair $\langle h : x \to y, k : y \to z\rangle$ (an element of the composite bimodule $(\iota_J)_\# \mathbb{A}_f$) to the composite $kh \in (\iota_I)_\#$.

3.1. Proposition. *The above diagram exhibits $\mathbb{A}$ as a **representable lax-colimit** of the lax functor Λ_a.*

Proof. The qualificative 'representable' means that the lax colimit property holds in the bicategory of bimodules with respect to lax cocones whose components are representable bimodules, with the uniquely determined mediating bimodule being representable as well (see (Hermida, 2001, §2.2) for the case of representable lax colimits of monads, which we also recall in §5.1).

Given such a lax cocone with components $\kappa_I : \mathbb{A}_I \to \mathbb{D}$ and $\lambda_f : (\kappa_J)_\# \mathbb{A}_f \Rightarrow (\kappa_I)_\#$, the unique mediating functor $\langle \kappa, \lambda \rangle : \mathbb{A} \to \mathbb{D}$ sends an object $x \in \mathbb{A}$ to $\kappa_{ax}(x)$. For its action on morphisms, regard a morphism $h : x \to y$ in $\mathbb{A}$ as an element of the bimodule $((\kappa_{ay})_\# \mathbb{A}_{ah})\,(x, y)$ (via the equivalence class of the pair $\langle h, id_y \rangle$), and let $\langle \kappa, \lambda \rangle(h) = \theta_{ah} \langle h, id_y \rangle$. □

Just like in the example of coverings of topological spaces, we can aggregate the 'transition morphisms' (either reindexing functors f^* or bimodules $\mathbb{A}_f$) together with their coherent associativities (isomorphisms $\delta_{f,g}$ above) by considering the diagram

$$\mathbb{B}^{\to} \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{B}$$

where the functors d and c take the domain and codomain respectively of a morphism (as an object of the arrow category $\mathbb{B}^{\to}$). The transition morphisms are grouped together into a single morphism $\theta : d^*\mathbb{A} \to c^*\mathbb{A}$, while their associativity data becomes a 2-cell

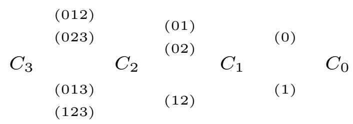
$$\delta : ((12)^*\theta)\,((01)^*\theta) \Rightarrow (02)^*\theta$$

where the functors $(01), (12), (02) : \mathbb{B}^{\to\to} \to \mathbb{B}^{\to}$ select the components and the composite, respectively, of a composable pair of arrows. The coherence conditions for δ become equations between the relevant restrictions to composable triples. Of course, we can now relativise these data to a 'covering' of $\mathbb{B}$, $q : \mathbb{T} \to \mathbb{B}$, as in Definition 3.2 below.

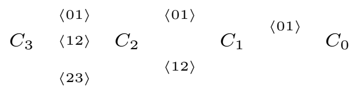
We hope that these considerations on how data such as (fibred) functors into a category are determined as lax colimits will guide the reader through our formalisation of pseudo-descent below.

3.1. The 2-category of descent data associated to a 3-truncated simplicial object

Given a 2-fibration $P : \mathcal{E} \to \mathcal{B}$, we consider a truncated simplicial object $\underline{C}$ in $\mathcal{B}$ of dimension 3:

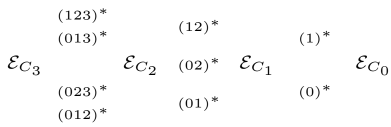


where $(ijk) : [3] \to [2]$ takes the vertices i, j, k to $0, 1, 2$ respectively and similarly for the remaining morphisms. In the opposite direction, we have

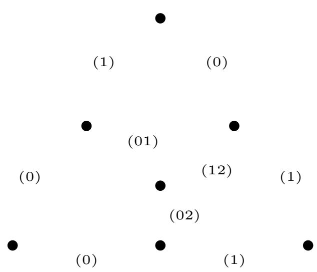


where $\langle ij \rangle : [n] \to [n+1]$ inserts an identity between i and j.

Such a complex induces, via $\mathcal{F}_P : \mathcal{B}^{coop} \to 2\text{-}\mathcal{C}at_{\otimes}$, the following diagram of 2-categories and 2-functors:

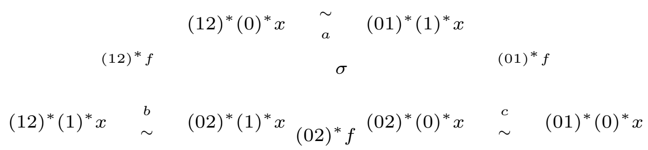


together with pseudo-natural isomoporphisms corresponding to the simplicial identities, *e.g.* $\sigma : (12)^*(0)^* \Rightarrow (01)^*(1)^*$, which we would not name explicitly, and invertible modifications between the relevant composites of such transformations. To keep track of the simplicial identities for the faces it helps to depict them in terms of *baricentric subdivisions* of the simplexes, *e.g.*



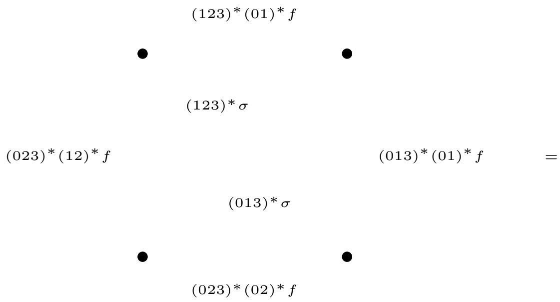
Now, we can consider the *pseudo descent object*[2] of the above diagram in 2-$\mathcal{C}at_\otimes$. The 2-category of (normalised) descent data $\mathsf{Des}(\underline{C})$:

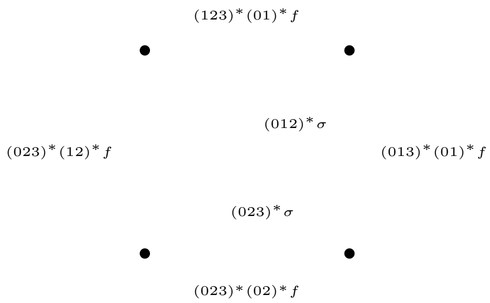
objects An object is a triple (x, f, σ), where x is an object of $\mathcal{E}_{C_0}$, $f : (1)^*x \to (0)^*x$ a morphism in $\mathcal{E}_{C_1}$ and σ is an invertible 2-cell in $\mathcal{E}_{C_2}$:



subject to the following axioms:

1. (commutative tetrahedron)





where we have subsumed (and left nameless) composites of coherent isomorphisms associated to various reindexings of the 2-simplex σ to simplify the diagram. For instance, the leftmost = is the composite

$$(023)^*(12)^*f\{\alpha\} \cong [(12)(023)]^*f\{\alpha\} = \{\beta\}[(02)(123)]^*f \cong \{\beta\}(123)^*(02)^*f$$

where $\alpha = (023)^*a$ and $\beta = (123)^*a$. Thus, the 'vertices' $\bullet$ above are these isomorphisms between 0-cells (we may regard them as

[2] This is a kind of weighted bilimit. There is an attendent theory of those for 2-$\mathcal{C}at_\otimes$ qua 2-$\mathcal{C}at_\otimes$-category. See (Lack, 2000) for some instances of *Gray*-limits

'thickened' 0-cells) and the = represent the corresponding 'pseudo-squares'.

2.

$$x \;\sim\; \langle 01\rangle^*(1)^*x \xrightarrow{\langle 01\rangle^* f} \langle 01\rangle^*(0)^*x \;\sim\; x \;=\; x \xrightarrow{id} x$$

3.

$$\langle 01\rangle^*\sigma \equiv id_f \quad \langle 12\rangle^*\sigma \equiv id_f$$

where we have simplified the expressions by putting $\equiv$ to indicate 'equality when pasted with the appropriate invertible modifications' as in the commutative tetrahedron above.

morphisms A morphism consists of a pair $(u, \mu) : (x, f, \sigma) \to (y, g, \gamma)$ where $u : x \to y$ in $\mathcal{E}_{C_0}$ and

$$\begin{array}{ccc} (1)^*x & \xrightarrow{f} & (0)^*x \\ {\scriptstyle (1)^*u}\downarrow & \mu & \downarrow{\scriptstyle (0)^*v} \\ (1)^*y & \xrightarrow[g]{} & (0)^*y \end{array}$$

is an invertible 2-cell in $\mathcal{E}_{C_1}$ satisfying the evident compatibility condition with σ and γ ('commutative prism' in $\mathcal{E}_{C_2}$).

2-cells A 2-cell $\theta : (u, \mu) \Rightarrow (v, \nu)$ is a 2-cell $\theta : u \Rightarrow v$ in $\mathcal{E}_{C_0}$ such that

$$g(u^*\theta)\circ\mu = \nu\circ(v^*\theta)f$$

Composition and identities in $\mathsf{Des}(\underline{C})$ are induced by those in $\mathcal{E}_{C_0}$, in the evident manner.

3.2. The 3-truncated simplicial object defined by a morphism and effective descent

Given a morphism $q : T \to Q$ in $\mathcal{B}$, we consider its comma-object

$$\begin{array}{ccc} q^{(2)} & \xrightarrow{d} & T \\ {\scriptstyle c}\downarrow & \lambda & \downarrow{\scriptstyle q} \\ T & \xrightarrow[q]{} & Q \end{array}$$

The top span defines a directed graph. Using the universal property of the comma-object, we endow it with identities and composition so as to make it an internal category (thus, the category structure is determined by that of the 2-cells with target Q, *cf.*(Street, 1973) for details). Let $\underline{\mathsf{Cat}}_q$ denote the 3-truncated simplicial object corresponding to the nerve of this internal category:

$$q^{(4)} \quad \begin{matrix} (012) \\ (023) \\ (013) \\ (123) \end{matrix} \quad q^{(3)} \quad \begin{matrix} (01) \\ (02) \\ (12) \end{matrix} \quad q^{(2)} \quad \begin{matrix} (d) \\ (c) \end{matrix} \quad T$$

so that $q^{(2)}$ is the object of 'pairs of composable 2-cells with target Q', $q^{(3)}$ is the object of composable triples of such and so on.

The morphism q induces a 2-functor $q^{(*)} : \mathcal{E}_Q \to \mathsf{Des}(\underline{\mathsf{Cat}}_q)$, taking an object x to the descent datum $(p^*x, \langle\lambda\rangle_x, \alpha_x)$, where $\langle\lambda_x\rangle : c^*p^*x \to d^*p^*x$ is the component of the pseudo-natural transformation[3] $\lambda^* : (pc)^* \Rightarrow (pd)^*$ at x pre- and post-composed with the isomorphisms $c^*p^*x \cong (pc)^*x$ and $d^*p^*x \cong (pd)^*x$. Because the internal-composition morphism $(02) : q^{(3)} \to q^{(2)}$ of $\underline{\mathsf{Cat}}_q$ is induced by the pasting composition of λ with itself, there is a canonical associativity modification

$$\alpha_x \equiv \delta_{\lambda,\lambda}(x) : [(01)^*\langle\lambda\rangle_x \cong (12)^*\langle\lambda\rangle_x] \triangleright [\cong (02)^*\langle\lambda\rangle_x \cong]$$

which completes our descent datum. The action on morphisms and 2-cells is similarly evident.

3.2. Definition. A morphism $p : T \to Q$ is of **effective descent** for $P : \mathcal{E} \to \mathcal{B}$ if the induced 2-functor $q^{(*)} : \mathcal{E}_Q \to \mathsf{Des}(\underline{\mathsf{Cat}}_q)$ is a biequivalence of 2-categories.

3.3. Remark. We may also consider *lax descent data* above, where the 2-cell σ is not required to be invertible. This is the formulation of higher descent advocated in (Street, 1995); it is the natural situation to consider when extending the data to higher-dimensions. However, the 2-categorical applications of descent which have appeared so far in the literature involve only pseudo-descent as above.

A far more restricted version of 2-descent is considered in (Duskin, 1989) in the setting of a *functor* $F : \mathbb{C}^{op} \to 2-\mathcal{C}at$, with $2-\mathcal{C}at$ considered merely as a category. The 2-dimensional descent data in *ibid.* agrees with our presentation above, ommiting all coherent isomorphisms. Of course, the considerations of 2-descent in *ibid.* are specifically oriented to the study of non-abelian cohomology for the topos $\mathbb{C}$.

4. Pseudo-descent vs. pseudo-monadicity

Given a fibration with sums $p : \mathbb{E} \to \mathbb{B}$, a morphism $q : \tilde{X} \to X$ induces a monad $T = q^*\Sigma_q : \mathbb{E}_{\tilde{X}} \to \mathbb{E}_{\tilde{X}}$. A classical result of (Bénabou and Roubaud, 1970) (independently credited to Beck) identifies T-algebras with descent data for q, $T - alg \simeq \mathsf{Des}(\underline{\mathsf{Cat}}_q)$. Consequently, q is of effective descent iff $q^* : \mathbb{E}_X \to \mathbb{E}_{\tilde{X}}$ is monadic.

[3] See (Hermida, 1999) for an explicit description of $\langle\lambda\rangle$ and the associativity modifications relating the composites of such $\langle\lambda\rangle$'s.

We set about establishing the corresponding result for pseudo-descent. Given a 2-fibration with direct images $P : \mathcal{E} \to \mathcal{B}$, a morphism $q : T \to Q$ induces a pseudo-monad $\mathsf{T} = q^*\Sigma_q : \mathcal{E}_T \to \mathcal{E}_T$. We refer to (Kelly, 1974; Lack, 2000; Marmolejo, 1999) for details on pseudo-monads and their pseudo-algebras. Given our simplyfing normalisation of descent data (conditions (2) and (3) in the definition) , we restrict our attention to *normal pseudo-algebras*, *i.e.* those which satisfy a strict unit axiom $a \circ \eta_A = id$ (where $a : \mathsf{T}A \to A$ is the algebra structure 1-cell).

Recall that $\underline{\mathsf{Cat}}_q$ is defined by the comma-object

$$\begin{array}{ccc} q^{(2)} & \xrightarrow{d} & T \\ {\scriptstyle c}\downarrow & \lambda & \downarrow{\scriptstyle q} \\ T & \xrightarrow[q]{} & Q \end{array}$$

and therefore $\Sigma_d c^* \simeq q^*\Sigma_q$. Given a descent datum (x, f, σ), we transpose f across $\Sigma_d \dashv d^*$ to obtain a morphism ${}^\bullet\hat{f} : q^*\Sigma_q x \simeq \Sigma_d c^* x \to x$, and we transponse σ across the biadjunction $\Sigma_d\Sigma_{(01)} \dashv (01)^*d^*$ to get a 2-cell $\hat{\sigma}$ with 0-codomain x and 0-domain $\Sigma_d\Sigma_{(01)}(12)^*c^*(x)$. Considering the pullback

$$\begin{array}{ccc} q^{(3)} & \xrightarrow{(01)} & q^{(2)} \\ {\scriptstyle (12)}\downarrow & & \downarrow{\scriptstyle c} \\ q^{(2)} & \xrightarrow[d]{} & Q \end{array}$$

Proposition 2.4 yields an equivalence $\Sigma_{(01)}(12)^* \simeq c^*\Sigma_d$, which together with the equivalence $q^*\Sigma_q \simeq \Sigma_d c^*$ yields an invertible 2-cell ${}^{\bullet\bullet}\hat{\sigma}$ with 0-domain $q^*\Sigma_q q^*\Sigma_q(x)$.

4.1. Lemma. *The assigment* $(x, f, \sigma) \mapsto (x, {}^\bullet\hat{f}, {}^{\bullet\bullet}\hat{\sigma})$ *extends to a 2-functor* $\hat{(\_)} : \mathsf{Des}(\underline{\mathsf{Cat}}_q) \to \mathsf{Ps}\text{-}T\text{-}\mathsf{alg}$.

Proof. The 2-cell ${}^{\bullet\bullet}\hat{\sigma}$ has the appropriate 1-domain and codomain, by the same argument as the 1-dimensional version (Beck-Bénabou-Roubaud theorem and Proposition 2.4). The remaining details involve tedious calculations. Let us just point out that the commutative tetrahedron axiom (1) yields the associativity axiom for the pseudo-algebra, while axiom (2) yields the unit conditions. □

4.2. Theorem. *The 2-functor* $\hat{(\_)} : \mathsf{Des}(\underline{\mathsf{Cat}}_q) \to \mathsf{Ps}\text{-}T\text{-}\mathsf{alg}$ *is a biequivalence.*

Proof. We construct a pseudo-inverse to $\hat{(\_)}$ by performing 'biadjoint transposition' in the opposite direction, thereby yielding equivalences when going around in both directions. We omit the laborious calculation. □

In Appendix A we provide a 'Beck style' criterion for the pseudo-monadicity of a 2-functor, *i.e.*, given $U : \mathcal{K} \to \mathcal{B}$ with a left biadjoint, when is it the case that $\mathcal{K}$ is the 2-category of pseudo-algebras for the pseudo-monad induced on $\mathcal{B}$?

4.3. Corollary. *A morphism $p : T \to Q$ is of effective descent iff $p^* : \mathcal{E}_Q \to \mathcal{E}_T$ is pseudo-monadic.*

5. Strong 2-regularity

5.1. General considerations on regularity

Let us recall that a regular category $\mathbb{R}$ (see (Freyd and Scedrov, 1990, 1.52) and (Borceux, 1994, §2)) is one which admits finite limits and pullback-stable coequalisers of kernel pairs. Some fundamental consequences of regularity are:

1. there is a regular-epi/mono factorisation system, and such factorisation is pullback-stable,
2. the fibration of subobjects $cod : Sub(\mathbb{R}) \to \mathbb{R}$ (the 'internal logic' of $\mathbb{R}$) admits direct-images (via the regular-epi/mono factorisation), satisfying the Beck-Chevalley condition,
3. there is an associated bicategory of relations $\mathcal{R}el(\mathbb{R})$, a relation being understood as a jointly monic span. Composition of such spans is defined via pullbacks and direct-images, the pullback-stability of the regular-epi/mono factorisation being equivalent to the associativity of the composition thus defined.
4. regular epis are of effective descent for the fibration of subobjects $cod : Sub(\mathbb{R}) \to \mathbb{R}$. Furthermore, if $\mathbb{R}$ is exact, regular epis are also of effective descent for the basic fibration $cod : \mathbb{R}^{\to} \to \mathbb{R}$.

We now seek a suitable collection of finite limits/colimits and exactness conditions in the 2-dimensional context which would entail anologous consequences to those listed above. It has long been known that a suitable 2-dimensional analogue of relation is the notion of *bimodule* or two-sided discrete fibration (Street, 1980). It follows from the considerations in *ibid.* that in order to have a 'calculus of bimodules' we require a 2-category $\mathcal{R}$ with comma-objects, pullbacks along (co)fibrations, and 'bounded' (bi)coinverters (see §5.2) stable under such pullbacks. We shall use the notational conventions and basic facts on bimodules from (Hermida, 2001, §2) (this reference contains a substantial application of bimodules in 2-dimensional categorical algebra).

As for the properties related to the basic fibration $cod : \mathbb{R}^{\to} \to \mathbb{R}$, we should recall that an object in the fibre $\mathbb{R}/I$ axiomatises the (Bourbaki version of the) notion of I-indexed family of objects. In the 2-dimensional situation, the corresponding concept is captured by the notion of *fibration* (the concept of family must incorporate an action of the 'morphisms' or generalised 2-cells of the object I on its fibres). Thus the role of the basic fibration is taken over by the basic 2-fibration $cod : \mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$, whose fibre over I is the 2-category $\mathcal{F}ib(\mathcal{R})/I$ of fibrations with base I. Here we assume the representable notion of fibration in a 2-category as advocated in (Street, 1973). The existence of pullbacks along (co)fibrations assures that cod is a 2-fibration indeed. The associated ‘fibration of subobjects’ is replaced by the sub-2-fibration of *discrete*[4] *fibrations*, as this notion of ‘predicate’ is consistent with the above replacement of jointly monic spans by bimodules. We will not be concerned with this latter sub-2-fibration however, as the results for it readily specialise from the more general ones for $cod : \mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$.

As we will show in §5.3 below, the assumptions on $\mathcal{R}$ above enabling a calculus of bimodules ensure that the basic 2-fibration admits direct-images, with the Beck-Chevalley condition over comma-objects.

As for the descent properties, we must identify a good candidate for ‘covering morphism’. We have already argued in §3 the relevance of lax colimits to (re)construct gadgets such as (fibration) functors. A morphism $q : O \to Q$ gives rise to a monad in $\mathsf{Bimod}(\mathcal{R})$, via the comma-object

$$\begin{array}{ccc} q^{(2)} & \xrightarrow{d} & O \\ {\scriptstyle c}\downarrow & \lambda & \downarrow{\scriptstyle q} \\ O & \xrightarrow[q]{} & Q \end{array}$$

since as we have pointed out in §3.1, the top-left span underlies an internal category $\underline{\mathcal{C}at}_q$. The inclusion of the category of bimodules into that of spans (for a given pair of objects, *e.g.* $J : \mathsf{Bimod}(\mathcal{R})(O,O) \to \mathbf{Spn}(\mathcal{R})(O,O)$) admits a reflection (Hermida, 2001, §2. (7)). Since the multiplication of the category $(O, q^{(2)})$ is consistent with bimodule composition ($(\langle fg, h\rangle \sim \langle f, gh\rangle \mapsto (fg)h = f(gh))$ the bimodule $\langle d, c\rangle\colon O \not\to O$ inherits a monad structure [5] from $\underline{\mathcal{C}at}_q$.

This monad bimodule $\langle d, c\rangle\colon O \not\to O$ is our 2-dimensional analogue of the kernel of a morphism, and we shall refer to it as the **lax-kernel monad** of q.

Similarly, instead of the coequaliser of a kernel pair, we consider the *representable Kleisli object* (in the sense of (Hermida, 2001, §2.2)[6]) of the monad $\langle d, c\rangle$, which is its lax (bi)colimit. More precisely, let $\mathbf{BiMod}_{\mathcal{R}}$ be the 2-category whose objects are endomorphisms in $\mathsf{Bimod}(\mathcal{R})$ with a monoid structure (monads), $M\colon X \not\to X$, whose morphisms $(f,\theta) : M \to N$ are pairs consisting of a 1-cell $f : X \to Y$ and a 2-cell

$$\begin{array}{ccc} X & \xrightarrow{M} & X \\ {\scriptstyle f_\#}\downarrow & \theta_f & \downarrow{\scriptstyle f_\#} \\ Y & \xrightarrow[N]{} & Y \end{array}$$

[4] Note that in the bicategorical context ‘discrete’ really means ‘groupoidal’, since the notion must be invariant under equivalence.

[5] Note that the bicategorical nature of the colimits involved in bimodule composition would only yield a pseudo-monad structure on $\langle d, c\rangle$, but this bimodule is genuinely discrete, hence the pseudo-monad structure on it becomes a monad.

[6] Here we consider the corresponding lax colimit in its bicategorical version, that is, uniquely characterised up to equivalence rather than up to isomorphism.

compatible with the monad structures on M,N, while 2-cells $\alpha : (f, \theta) \Rightarrow (f', \theta')$ are 2-cells $\alpha : f \Rightarrow f'$ compatible with θ and θ'. Forgetting the bimodules we get a domain/codomain object functor $U : \mathbf{BiMod}_{\mathcal{R}} \to \mathcal{R}$ which has a left section $\mathsf{Hom} : \mathcal{R} \to \mathbf{BiMod}_{\mathcal{R}}$ sending an object X to its Hom-bimodule $\mathsf{Hom}_X : X \not\rightarrow X$ (whose top object is the object-of-arrows $X^{\to}$). The existence of representable Kleisli objects for monads in $\mathsf{Bimod}(\mathcal{R})$ amounts to the existence of a *left biadjoint* $\mathcal{Q} \dashv \mathsf{Hom}$, in which case we get a 1-cell $q : X \to \mathcal{Q}(M)$ and a 2-cell

$$\begin{array}{ccccc} & & M & & \\ X & & & & X \\ & & \tau & & \\ q_{\#} & & \mathcal{Q}(M) & & q_{\#} \end{array}$$

forming a lax cocone for the unit and multiplication of the monad M and universal among such.

5.1. Definition. We say that $q : O \to Q$ is **essentially-surjective-on-objects** (*eso* for short) if the comma-object

$$\begin{array}{ccc} q^{(2)} & d & O \\ c & \lambda & q \\ O & q & Q \end{array}$$

exhibits (q, λ) as the representable Kleisli object of the monad $(d, c) \colon O \not\rightarrow O$

We remind the reader that in $\mathcal{C}at$ this notion agrees with the usual one, *i.e.* a functor $q : \mathbb{O} \to \mathbb{Q}$ such that every object x in $\mathbb{Q}$ is isomorphic to the image of some object o of $\mathbb{O}$, $qo \cong x$. The above characterisation of an eso as a lax colimit is taken from (Street and Walters, 1978, p.369). The fact that we take our lax colimit in the bicategorical sense (unique up to equivalence rather than isomorphism) means that we weaken the characterisation in *ibid.* from *bijective on objects* to *essentially surjective* functors.

5.2. Strongly 2-regular 2-categories

We recall from (Kelly, 1989) the notions of coinserter, coequifier and coinverter in a 2-category $\mathcal{K}$, in their bicategorical versions, as bicolimits:

– Given a parallel pair of morphisms $f, g : X \to Y$, their *coinserter* consists of a morphism $h : Y \to C$ and a 2-cell $\theta : hf \Rightarrow hg$ universal among such: composition with h induces an equivalence of categories

$$\mathcal{K}(C, A) \simeq 2 - \mathsf{cells}((f, g), A)$$

pseudo-natural in A, where the right-hand side category consists of pairs $(z : Y \to A \ , \gamma : zf \Rightarrow zg)$ as objects and the evident compatible 2-cells between such as morphisms. In $\mathcal{C}at$, such a coinserter can be computed by *generators and relations*, exploiting the monadicity of categories over graphs. See (Gabriel and Zisman, 1967).

- Given a parallel pair of morphisms $f, g : X \to Y$ and a parallel pair of 2-cells between them, $\alpha, \beta : f \Rightarrow g$, their *coequifier* consists of a morphism $s : Y \to Q$ such that $s\alpha = s\beta$ and universal among such. In $\mathcal{C}at$, the category Q is obtained by quotienting Y by the congruence generated by the relation $\alpha_x \equiv \beta_x$ for all $x \in X$.
- Given a parallel pair of morphisms $f, g : X \to Y$ and a 2-cell between them $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ (α), its *coinverter* is a morphism $q : Y \to Y[\alpha^{-1}]$ such that $q\alpha$ is an isomorphism, and universal among such. Once again, in $\mathcal{C}at$ the construction can be carried out by generators and relations, as for the colimits above (Gabriel and Zisman, 1967).

We will refer to a *bounded coinverter* when the 2-cell $\alpha : f \Rightarrow g$ to be coinverted is already provided with a morphism $b : Y \to Z$ such that $b\alpha$ is an isomorphism. We apply a similar convention for the other kind of colimits above and representable Kleisli objects, so that the representable Kleisli object of a lax-kernel monad is a bounded such.

If $\mathcal{K}$ admits such bicolimits, so does any slice $\mathcal{K}/X$ (where they are created by the forgetful 'domain' 2-functor into $\mathcal{K}$). The reason why we insist on the bicategorical versions is that these are more commonly available than the strict 2-categorical ones *cf.*(Blackwell et al., 1989). The same can be said about pullbacks, but we only take pullbacks along (co)fibrations, and these are equivalent to 'pseudo-pullbacks' (Joyal and Street, 1993), which are flexible (just like comma-objects) and hence have both the 2-categorical and bicategorical universal properties.

5.2. Definition. A 2-category $\mathcal{R}$ is **strongly 2-regular** if

1. it admits pullbacks along (co)fibrations and comma-objects,
2. it admits bounded coinserters and coequifiers,
3. bounded coinserters and coequifiers are stable under pullbacks along (co)fibrations, *i.e.* for any (co)fibration $p : E \to B$, $p^* : \mathcal{R}/B \to \mathcal{R}/E$ preserves them.

We refer to bounded coinserters and coequifiers as $\Downarrow$-*colimits* in order to simplify statements about the consequences of the axioms. As we have already mentioned, the above structure is enough to allow the construction of a bicategory of bimodules $\mathsf{Bimod}(\mathcal{R})$. In §5.3 we show that they endow the basic fibration $cod : \mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$ with direct-images satisfying the Beck-Chevalley condition on comma-objects.

5.3. Examples.

1. $\mathcal{C}at$ is strongly 2-regular: the stability of $\Downarrow$-colimits follows easily from the fact that a (co)fibration p is *exponentiable* ((Giraud, 1964)), *i.e.* p^* has a right biadjoint, and thus preserves all bicolimits.
2. Given a category $\mathbb{R}$ with pullbacks, pullback-stable finite coproducts and coequalisers, and (cartesian) free monoids (*e.g.* a topos with a natural numbers object), the 2-category of internal categories $\mathcal{C}at(\mathbb{R})$ is strongly 2-regular. Bounded coinserters are built by means of coproducts and coequalisers (by 'generators and relations', exploiting the monadicity of categories over graphs).
3. If $\mathcal{R}$ is strongly 2-regular, so is $\mathcal{R}^{co}$. This is reason why we postulate the stability axioms for both fibrations and cofibrations.

5.4. Remark. If the 2-category $\mathcal{R}$ is locally discrete, that is, it is an ordinary category, we get the following degenarate instances of the concepts above:

– any morphism is a (covariant) fibration, since its codomain is discrete.
– comma-objects reduce to pullbacks
– coinserters become coequalisers, while coidentifiers are trivial, since any pair of parallel 1-cells related by a 2-cell must be equal.

Consequently, a locally discrete 2-category $\mathcal{R}$ is strongly 2-regular iff

– it admits pullbacks
– every slice $\mathcal{R}/X$ has coequalisers (this is the boundedness condition), preserved by pullback functors between slices.

Since these conditions are slightly more than those required for regularity (namely, stable coequalisers) we qualified our 2-dimensional version *strong*. In this context, the basic 2-fibration is the usual codomain fibration, the esos are coequalisers of their kernel pairs, and the descent result of Theorem 5.13 is that quoted in the introduction.

We point out some immediate consequences of the axioms for strong 2-regularity which we require later.

5.5. Proposition.

1. *Bounded coinverters exist and are stable under pullback along (co)fibrations.*
2. *Representable Kleisli objects for lax-kernel monads exist and are stable under pullback along (co)fibrations.*
3. *For a given a morphism $f : X \to Y$, the 2-functor $f^*_{\downarrow}(\_) : \mathcal{R}/Y \to \mathcal{R}/X$ taking a morphism $p : E \to Y$ to its projection from the comma-object $f \downarrow p$, preserves $\Downarrow$-colimits.*

Proof.
<u>(1)</u>: the coinverter of $\alpha : f \Rightarrow g$ is easily constructed from coinserters and

coequifiers: we coinsert a 2-cell α' between g and f, and force it to be the inverse of α via two coequifiers which impose the isomorphism equations.
(2): see (Hermida, 2001, Prop.2.6).
(3): We decompose the comma-object as follows

$$\begin{array}{ccc} f\!\downarrow\! p & \xrightarrow{f^*_{\downarrow}p} & X \\ & & \downarrow f \\ E & \xrightarrow[p]{} & Y \end{array} \quad = \quad \begin{array}{ccccc} f\!\downarrow\! p & \xrightarrow{(\pi'_f)^*p} & f\!\downarrow\! Y & \xrightarrow{\pi_f} & X \\ & & \downarrow \pi'_f & & \downarrow f \\ E & \xrightarrow[p]{} & Y & \xrightarrow[id]{} & Y \end{array}$$

where the square is a pullback. We thus factorise our 2-functor as $f^*_{\downarrow} \cong \Sigma_{\pi_f}(\pi'_f)^*$. The 2-functor $(\pi'_f)^*$ preserves $\Downarrow$-colimits because π'_f is a (free) cofibration. Since Σ_{π_f} is a left 2-adjoint, the composite $\Sigma_{\pi_f}(\pi'_f)^*$ preserves $\Downarrow$-colimits. □

5.6. Corollary. *Given a strongly 2-regular $\mathcal{R}$, for every object X, the 2-categories $\mathcal{F}ib(\mathcal{R})/X$ and $\mathcal{C}o\mathcal{F}ib(\mathcal{R})/X$ of internal (co)fibrations satisfy the axioms of strong 2-regularity.*

Proof. As we have already pointed out, the slices $\mathcal{R}/X$ have the relevant limits and colimits. Since (co)fibrations are pseudo-algebras over $\mathcal{R}/X$ for the monad $X \downarrow (\_)$ (and its dual), they inherit all flexible limits and $\Downarrow$-colimits by Proposition 5.5.(3). The latter are henceforth pullback stable along (co)fibrations. □

5.7. Remark. The above result ('$\mathcal{R}$ strongly 2-regular implies $\mathcal{F}ib(\mathcal{R})/X$ strongly 2-regular') is the proper 2-dimensional counterpart of the slicing result for regular categories ('$\mathbb{R}$ regular implies $\mathbb{R}/X$ regular'), since fibrations play the role of 'morphisms as families' in the present context.

Recall (Borceux, 1994, §2.6) that a regular category is *exact* if every equivalence relation is effective (that is, it has a quotient and is the kernel pair of its quotient map). In our 2-dimensional examples, $\mathcal{C}at$ could be considered '2-exact', since monads in $\mathsf{Bimod}(\mathcal{C}at)$ admit representable Kleisli objects and they become lax-kernel monads (see (Hermida, 2001, §2.1)), pullback stable along (co)fibrations. The same property applies to the fibrational slices $\mathcal{F}ib(\mathcal{R})/X$ of such a '2-exact' $\mathcal{R}$.

Coinserters and coidentifiers from Kleisli objects and coinverters
The two main consequences of strong 2-regularity which we will show are the construction of direct-images for the basic fibration (Proposition 5.9) and the effective descent property of esos (Theorem 5.13). The former requires only bounded coinverters, while the second requires the construction of bounded pseudo-coequalisers, which could achieved via bounded representable Kleisli objects. Hence, it is reasonable to ponder whether the axiomatics should require these two kind of colimits rather than coinserters and coequifiers. The

point is that these latter are more primitive, as they enable us to construct the others via generators and relations, as shown above. But in the presence of additional completeness conditions, we could also obtain coinserters and coequifiers from the other kind of colimits above:

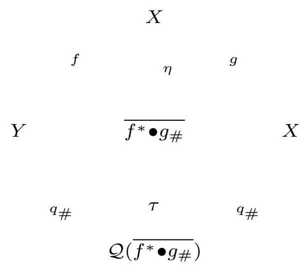
5.8. Proposition.

1. *A 2-category $\mathcal{R}$ which admits coinverters and local (co/)equalisers, meaning every hom-category does, it admits coequifiers as well.*
2. *A 2-category $\mathcal{R}$ which :*
   - *admits a calculus of bimodules (pullbacks of (co)fibrations, comma-objects and stable bounded coinverters),*
   - *admits the construction of free monads on endobimodules,*
   - *and (bounded) representable Kleisli objects,*

   also admits (bounded) coinserters

Proof.
(1): To construct the coequifier of $\alpha, \beta : f \Rightarrow g : X \to Y$, consider its equaliser $\epsilon : e \Rightarrow f$ in $\mathcal{R}(X, Y)$, and compute the coinverter of the resulting single 2-cell $\theta = \alpha\epsilon = \beta\epsilon$, $q : Y \to Y[\theta^{-1}]$. This is the required coidentifier, since any morphism which coidentifies α and β must coinvert ϵ and viceversa.
(2): Let us recall the 1-dimensional analogue of this result: to compute the coequaliser of a pair of morphisms $f, g : X \to Y$ in $\mathcal{S}et$, we take their image $\mathsf{Im}\langle f, g\rangle$ in $Y \times Y$ and consider the equivalence relation which it generates, $\overline{\mathsf{Im}\langle f, g\rangle}$. Finally, we obtain the coequaliser as the quotient by this equivalence relation, $q : Y \to Y/\overline{\mathsf{Im}\langle f, g\rangle}$.

We repeat this process working with bimodules rather than relations: given $f, g : X \to Y$, we consider the 'image' bimodule $f^* \bullet g_\#$, compute the free monad on it, $\overline{f^* \bullet g_\#}$ and consider its representable Kleisli object:



The resulting 2-cell $\tau\eta : qf \Rightarrow qg$ is the coinserter of f, g.

$\square$

For instance in $\mathcal{C}at(\mathcal{E})$, for $\mathcal{E}$ a topos with a natural numbers object, we could use the second construction above to compute coinserters. The construction becomes particularly simple if the bimodule $f^* \bullet g_\#$ already bears a monad structure, as it is the case when the pair f, g forms the (domain,codomain) data of a truncated 3-simplicial object. Notice that (Hermida, 2001, Prop. 8.3) shows that the construction of representable Kleisli objects for $\mathcal{C}at(\mathcal{E})$ is fairly easy, requiring only pullbacks in the ambient category $\mathcal{E}$.

5.3. Direct images for the basic 2-fibration

We show that in a strongly 2-regular 2-category $\mathcal{R}$, the basic 2-fibration $cod : \mathcal{F}ib(\mathcal{R}) \to R$ admits direct images.

Consider a fibration $p : \mathbb{E} \to \mathbb{B}$ in $\mathcal{C}at$ and let $B : \mathbb{B} \to \mathbf{1}$ be the unique functor into the terminal category. Any category $\mathbb{D}$ yields a fibration over $\mathbf{1}$, $D : \mathbb{D} \to \mathbf{1}$; a morphism in $\mathbb{D}$ is cartesian iff it is an isomorphism. Thus if we have a morphism in $\mathcal{F}ib(\mathcal{K})$

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{H} & \mathbb{D} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle D} \\ \mathbb{B} & \xrightarrow[B]{} & \mathbf{1} \end{array}$$

the functor $H : \mathbb{E} \to \mathbb{D}$ must take the cartesian morphisms of $\mathbb{E}$ to isomorphisms in $\mathbb{D}$. Thus $\Sigma_B(p) = \mathbb{E}[cart^{-1}]$, the category of fractions in which we universally invert the cartesian morphisms. This amounts to forcing the morphism in $\mathcal{K}^{\to}$

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{id} & \mathbb{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle E} \\ \mathbb{B} & \xrightarrow[B]{} & \mathbf{1} \end{array}$$

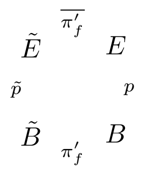
to become a morphism in $\mathcal{F}ib(\mathcal{K})$ by (co)inverting the canonical comparison between the cartesian morphisms of $\mathbb{E}$ (fibred over p) and those of $\mathbb{E}$ (fibred over $\mathbf{1}$). We give a more precise formulation below.

A similar argument works when taking direct images along a fibration $q : B \to B'$: the composite qp is a fibration and we force $(q, id) : p \to qp$ to be a morphism in $\mathcal{F}ib$ by universally inverting the relevant comparison 2-cell.

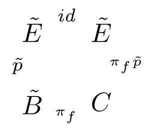
For the general construction, given a morphism $f : B \to C$ and a fibration $p : E \to B$ consider the comma-object

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\pi_f} & C \\ {\scriptstyle \pi'_f}\downarrow & \lambda & \downarrow{\scriptstyle id} \\ B & \xrightarrow[f]{} & C \end{array}$$

so that π_f is a (free) fibration, $f = \pi_f \eta_f$ and $\pi'_f \dashv \eta_f$. This latter adjunction induces a biadjunction $(\pi'_f)^* \dashv \eta_f^*$. Therefore, $\Sigma_f \equiv \Sigma_{\pi_f}(\pi'_f)^*$. Thus, we are reduced to computing the left biadjoint Σ_{π_f} along a (free) fibration. We take the pullback

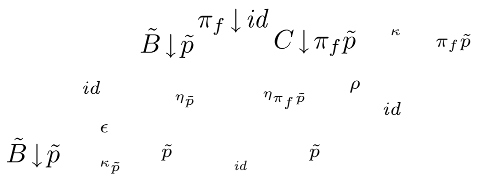


so that we have a morphism in $\mathcal{K}^{\rightarrow}$

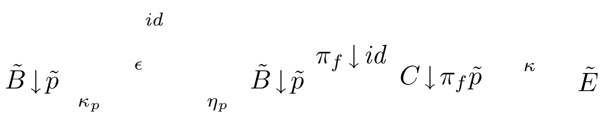


where the composite $\pi_f\tilde{p}$ is a fibration. Now we can ask for this morphism to preserve cartesiannes. Cartesian liftings are given by the counit of the adjunctions which characterise these fibrations internally. Let $\epsilon : \eta_{\tilde{p}}\kappa_{\tilde{p}} \Rightarrow id : \tilde{B}\downarrow\tilde{p} \rightarrow \tilde{B}\downarrow\tilde{p}$ be the counit ('cartesian morphisms' of $\tilde{E}$) and $\rho : id \Rightarrow \kappa\eta_{\pi_f\tilde{p}} : \pi_f\tilde{p} \rightarrow \pi_f\tilde{p}$ the (invertible) unit of the corresponding adjunction of the composite fibration.

The pair (π_f, id) is a morphism of fibrations if the following 'adjoint transpose' is an isomorphism (writing $\pi_f\downarrow id$ for the morphism between comma-objects canonically induced by the above morphism in $\mathcal{K}^{\rightarrow}$):



a condition which we enforce by taking $q : \tilde{E} \rightarrow \tilde{E}[({}^{\bullet}\epsilon)^{-1}]$ to be the coinverter of



This coninverted 2-cell lives in the slice $\mathcal{K}/C$ and thus we have an induced morphism $\Sigma_f(p) : \tilde{E}[({}^{\bullet}\epsilon)^{-1}] \rightarrow C$.

5.9. Proposition. *The assignment $p \mapsto \Sigma_f(p)$ is the object part of a left biadjoint to f^*, which endows the basic 2-fibration $cod : \mathcal{F}ib(\mathcal{K}) \rightarrow \mathcal{K}$ with direct images.*

Proof. The first point to make is that $\Sigma_f(p) : \tilde{E}[({}^{\bullet}\epsilon)^{-1}] \rightarrow C$ is indeed a fibration since the monad $B\downarrow\_ = id^*_{\downarrow} : \mathcal{K}/B \rightarrow \mathcal{K}/B$ preserves bounded coinverters (by Proposition 5.5.(3)) and therefore the algebra structures of $\pi_{\tilde{p}}$ and $\pi_f\tilde{p}$ induce one on $\Sigma_f(p)$.

Consider the diagram of pullbacks

$$\begin{array}{ccccc} & & id & & \\ E & \xrightarrow{\overline{\eta_f}} & \tilde{E} & \xrightarrow{\overline{\pi'_f}} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\eta_f} & \tilde{B} & \xrightarrow{\pi'_f} & B \\ & & id & & \end{array}$$

Since η_f is right-adjoint right-inverse to π'_f, so is $\overline{\eta_f}$ with respect to $\overline{\pi'_f}$ by (Hermida, 1999, Lemma 4.1).

The unit of the biadjunction is then determined by the following morphism in $\mathcal{F}ib(\mathcal{K})$

$$\begin{array}{ccccccc} E & \xrightarrow{\overline{\eta_f}} & \tilde{E} & \xrightarrow{id} & \tilde{E} & \xrightarrow{q} & \tilde{E}[({}^{\bullet}\epsilon)^{-1}] \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle \pi_f\tilde{p}} & \swarrow{\scriptstyle \Sigma_f(p)} & \\ B & \xrightarrow{\eta_f} & \tilde{B} & \xrightarrow{\pi_f} & C & & \end{array}$$

For stability, consider a comma-object

$$\begin{array}{ccc} B' & \xrightarrow{x'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{x} & C \end{array} \quad = \quad \begin{array}{ccccc} B' & \xrightarrow{z} & \tilde{B} & \xrightarrow{\pi'_f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{x} & C & \xrightarrow{id} & C \end{array}$$

which we have rewritten in terms of a 'free-fibration' comma-object pasted with a pullback square. Once again, we can factor $x = \eta_x\pi'_x$ where π'_x is the free cofibration and η_x has a right adjoint π_x. Hence pulling back along x, $x^* \cong \eta_x^*(\pi'_x)^*$, preserves the coinverter on $\pi_f\tilde{p}$ defining the direct image (pulling back along a cofibration does, and pulling back along η_x has a right biadjoint π_x^* when restricted to the full sub-2-category of the slices spanned by the fibrations). We thus get an appropriate coinverter on $x^*(\pi_f\tilde{p}) \cong f'\circ(z^*\tilde{p})$, which is the one defining $\Sigma_{f'}((x')^*p)$ since f' is a fibration.

□

5.4. Effective descent for an eso

Given a strongly 2-regular 2-category $\mathcal{R}$, we want to show that an eso is of effective descent for the basic 2-fibration $cod : \mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$. We do so via the pseudo-monadicity Theorem A.2, by appealing to the identification of descent data with pseudo-algebras.

The notion of fully-faithfulness for a morphism $f : X \to Y$ in $\mathcal{K}$ is the representable one (for every Z, $\mathcal{K}(Z, f) : \mathcal{K}(Z, X) \to \mathcal{K}(Z, Y)$ is fully-faithful). Just like in $\mathcal{C}at$, a morphism e in $\mathcal{K}$ is an equivalence iff it is eso and fully-faithful. The reason is the following orthogonality property (*cf.* (Street and Walters, 1978, Prop. 23)):

5.10. Proposition. *Essentially-surjective-on-objects are essentially orthogonal to fully faithful morphisms: given a pseudo-commutative diagram as on the left, it can be factorised as on the right*

$$\begin{array}{ccc} O & \xrightarrow{q} & Q \\ {\scriptstyle a}\downarrow & \overset{\theta}{\cong} & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array} \quad = \quad \begin{array}{ccc} O & \xrightarrow{q} & Q \\ {\scriptstyle a}\downarrow & \overset{\alpha}{\cong}\ {\scriptstyle d}\swarrow\ \overset{\beta}{\cong} & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where q is eso and f is fully-faithful and the diagonal d is essentially unique up to a unique isomorphism (compatible with α and β)

Proof. Given $\sigma : qd \Rightarrow qc$ corresponding to the comma-object defining the eso q, the 2-cell $\theta \circ b\sigma \circ \theta^{-1} : fad \Rightarrow fac$ corresponds by fully-faithfulness of f to a unique $\sigma' : ad \Rightarrow ac$ and the bicategorical universality of (q, σ) as a lax colimit yield the required d,α and β. □

Thus, if e is eso and fully faithful, it is essentially orthogonal to itself, and the diagonal factorisation (as above) yields a pseudo-inverse to e. We need also the following cancellation property of esos, whose proof is entirely analogous to that of the same cancellation property for regular epis:

5.11. Proposition. *Given composable morphisms $p : T \to D$ and $q : D \to Q$, if qp and p are esos, so is q.*

5.12. Remark. When the 2-category $\mathcal{R}$ is locally discrete, a morphism is fully faithful iff it is a monomorphism. Therefore, the above essential orthogonality result reduces to the usual orthogonality of regular epis and monos. Likewise for the cancellation property of esos reduces to the same one for regular epis.

5.13. Theorem. *In a strongly 2-regular 2-category $\mathcal{R}$, an essentially-surjective-on-objects morphism is of effective descent for the basic 2-fibration cod : $\mathcal{F}ib(\mathcal{R}) \to \mathcal{R}$.*

Proof. By Proposition 5.9, *cod* has direct images and we can therefore apply Theorem 4.2. We must therefore show that for an eso $q : O \to Q$, the change-of-base 2-functor $q^* : \mathcal{F}ib(\mathcal{K})/Q \to \mathcal{F}ib(\mathcal{K})/O$ is pseudo-monadic. According to Theorem A.2, we achieve this by showing the following:

– $\mathcal{F}ib(\mathcal{K})/Q$ has pseudo-coequalisers, since these can be built from (bounded) coinserters and coequifiers. These in turn are preserved by q^* by the same argument as the one given in the proof of stability in Proposition 5.9: factorise q as $q = \eta_q \pi'_q$, where π'_q is the free cofibration on q and η_q has π_q as right-adjoint. Since $\eta_q^* \dashv \pi_q^*$ is a biadjunction between the 2-categories of fibrations, q^* preserves pseudo-coequalisers.

– q^* reflects equivalences: given a morphism $e : p \to p'$ in $\mathcal{F}ib(\mathcal{K})/Q$ such that q^*e is an equivalence:

$$\begin{array}{ccc} q^*p & \xrightarrow{q_p} & p \\ {\scriptstyle q^*e}\downarrow & & \downarrow{\scriptstyle e} \\ q^*p' & \xrightarrow[q_{p'}]{} & p' \end{array}$$

In the pullback diagram above, both q_p and $q_{p'}$ are eso's since they are pullbacks of q along fibrations. Since q^*e is an equivalence, it is an eso, and therefore so is e: eq_p and q_p eso imply e eso by Proposition 5.11. It remains to show that e is fully faithful. Consider a pair of morphisms $a, b : x \to p$ in $\mathcal{F}ib(\mathcal{K})/Q$ and a 2-cell $\sigma : ea \Rightarrow eb$. We get a 2-cell $q^*\sigma : q^*(ea) \Rightarrow q^*(eb)$ and since $q^*(e)$ is fully faithful, there is a unique $\sigma' : q^*(a) \Rightarrow q^*(b)$ such $q^*(e)\sigma' = \sigma$. We thus have a 2-cell $q_p\sigma' : aq_x \Rightarrow bq_x$, and by the 2-dimensional universal property of the eso q_x we get a unique 2-cell $\hat{\sigma} : a \Rightarrow b$ such $\hat{\sigma}q_x = q_p\sigma'$. Also by universality $e\sigma' = \sigma$, and we conclude that e is fully faithful, and therefore an equivalence.

□

Appendix

A. Pseudo-monadicity for a 2-functor

Given a pseudo-functor $U : \mathcal{L} \to \mathcal{K}$ with a left biadjoint, *i.e.* a pseudo-functor $F : \mathcal{K} \to \mathcal{L}$, pseudo-natural trasnformations $t : id \Rightarrow UF$ and $w : FU \Rightarrow id$ and invertible modifications $\tau : id \triangleright Uw \circ tU$ and $\omega : wF \circ Ft \triangleright id$ satisfying the attendant axioms, we get a pseudo-monad $T = UF$ (we leave the remaining data implicit), and consequently a comparison pseudo-functor $K : \mathcal{L} \to \mathsf{Ps}\text{-}T\text{-}\mathsf{alg}$, $K(L) = (UL, Uw_L, Uw_{w_L}, \tau_L)$.

The pseudo-functor U is called **pseudo-monadic** when K is a biequivalence.

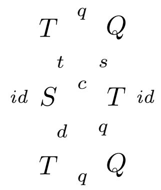
We follow our notational conventions in (Hermida, 2001) for pseudo-algebras.

A.1. Remark (Coherence). We simplify our statements and calculations appealing to coherence: we regard both U and F as 2-functors. Recall that this is achieved by the bireflection of 2-categories and pseudo-functors into 2-categories and 2-functors.

Recall from (MacLane, 1998) that the classical Beck monadicity theorem asks for a right adjoint U to create coequalisers of pairs of morphisms taken to a U*-split fork*. A split fork is given by the following data

$$S \;\overset{d}{\underset{c}{\rightrightarrows}}\; T \;\overset{q}{\rightarrow}\; Q \qquad t : T \to S, \quad s : Q \to T$$

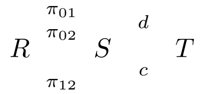
such that the following commutes



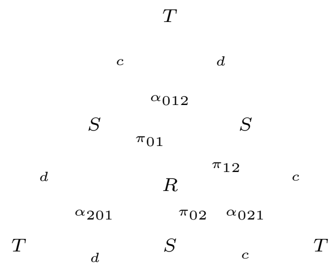
We have purposely drawn the diagrams as *naturality squares* so as to motivate the 2-dimensional version which follows.

A **2-fork** consists of

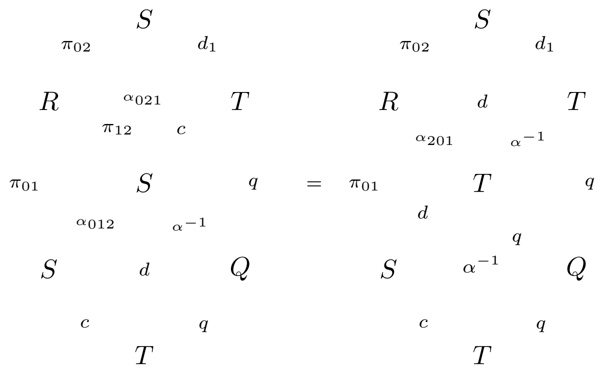
- 1-cells as displayed



- invertible 2-cells α's as displayed



A **pseudo-coequaliser** for a 2-fork consists of a pair (q, α), where $q : T \to Q$ and $\alpha : qd \Rightarrow qc$ is an invertible 2-cell, such that pasting the appropriate 3 copies on the above subdivided 2-simplex yields a commutative cube:



and universal (up to equivalence) among such data (bicolimit property). We may regard the cube as a pseudo-natural transformation from the (diagram of the) top-face to the bottom one. We now seek a **pseudo-splitting** of this cube qua pseudo-natural transformation: it consists (in addition to the given pair (q, α) satisfying the equations above) of morphisms $u : S \to R$, $t : T \to S$ and $s : Q \to T$, invertible 2-cells $\theta : ct \Rightarrow sq$, $\gamma : \pi_{12}u \Rightarrow tc$ and $\delta : \pi_{02}u \Rightarrow td$ satisfying the equations to yield a cube as above, and invertible 2-cells $\nu_R : id \Rightarrow \pi_{01}u$, $\nu_S : id \Rightarrow dt$ and $\nu_T : id \Rightarrow qs$ satisfying the equations of a modification.

A tedious calculation shows that a pseudo-split 2-fork is a pseudo-coequaliser, evidently an absolute one (preserved by homomorphisms of bicategories). Following the terminology of the 1-dimensional situation, a fork in $\mathcal{L}$ is called **U-pseudo-split** if it its image under U has a pseudo-splitting in $\mathcal{K}$.

Before formulating the pseudo-monadicity theorem, let us make explicit the relationship between pseudo-algebras and pseudo-split 2-forks: given a pseudo-algebra $x : TX \to X$ with structural isomorphisms 2-cells $\iota : id \Rightarrow x \circ \eta_X$ and $\mu : x \circ Tx \Rightarrow x \circ \mu_X$ yields a 2-fork

$$T^3X \quad \overset{\mu_{TX}}{\underset{T^2x}{T\mu_X}} \quad T^2X \quad \overset{Tx}{\underset{\mu_X}{}} \quad TX$$

where the objects stand for the free T-algebras on them and the structural 2-cells are given by the pseudo-naturality of μ and the invertible modification for pseudo-associativity. This 2-fork becomes U-pseudo-split, with the pseudo-coequalising data $(x : TX \to X, \mu : x \circ Tx \Rightarrow x \circ \mu_X)$ and pseudo splitting given by instances of η (η_X, η_{TX} and η_{T^2X}) and structural 2-cells given by the pseudo-naturality of η (the axioms for pseudo-algebras yield the commutative cube axioms) and the modification relating the splitting to the given 2-fork given by the obvious modification stating the unit axioms for the pseudo-monad.

A.2. Theorem (Pseudo-monadicity). *Given a 2-functor $U : \mathcal{L} \to \mathcal{K}$ with a left biadjoint (say F), the canonical comparison 2-functor $K : \mathcal{L} \to$ Ps-T-alg is a biequivalence iff the following hold*

1. *$\mathcal{L}$ has pseudo-coequalisers of U-pseudo-split 2-forks*
2. *U preserves such pseudo-coequalisers*
3. *U reflects equivalences*

Proof. The proof mimics the 1-dimensional Beck monadicity theorem, as given in (MacLane, 1998). We give only an outline. We write T for the 2-functor of the pseudo-monad UF.

The only if direction is quite straightforward: given a U-pseudo-split 2-fork, applying T to it we get another such, and universality yields the required pseudo-algebra structure to produce the required pseudo-coequaliser in Ps-T-alg, which is evidently preserved by U. The fact that U reflects equivalences is equally simple to establish *cf.*(Kelly, 1974).

In the converse direction, we construct a pseudo-inverse K' : Ps-T-alg $\to \mathcal{L}$ to the comparison 2-functor K as follows: given a pseudo-algebra $x : TX \to X$ with structural isomorphisms 2-cells $\iota : id \Rightarrow x \circ \eta_X$ and $\mu : x \circ Tx \Rightarrow x \circ \mu_X$ we obtain the following 2-fork in $\mathcal{L}$

$$FUFUFX \overset{FUFx}{\underset{w_{FUFX}}{FUw_{FX}}} FUFX \quad \overset{Fx}{\underset{w_{FX}}{}} \quad FX$$

with structural 2-cells given by the pseudo-naturality of w which becomes U-split as we indicated before stating the theorem. The target object of its pseudo-coequaliser gives the value of the required pseudo-inverse K'. □

A.3. Remark. A referee brought to our attention (Le Creurer et al., 2002), which is devoted to the proof of a Beck style pseudo-monadicity theorem (Theorem 3.6): a 2-functor $U : \mathcal{A} \to \mathcal{C}$, with a 2-functor pseudo-adjoint F, is pseudo-monadic iff U reflects adjoint equivalences, $\mathcal{A}$ has pseudo-coequalisers of U-absolute codescent objects and U preserves them.

Modulo the mild variation in terminology (codescent object = 2-fork), this theorem is an immediate consequence of ours, since a U-pseudo-split 2-fork is U-absolute, *i.e.* its U-image is absolute in $\mathcal{C}$, as we have already mentioned. Thus our formulation is a more accurate 2-dimensional version of Beck's criterion for monadicity and, in principle, more easily applicable because, when proving the existence of pseudo-coequalisers in $\mathcal{A}$, we have an explicit U-pseudo-splitting available. Nevertheless, both theorems apply equally well in our proof of effective descent for an eso. The reader may wish to consult *ibid.* for fairly detailed calculations, which we have ommitted from our brief presentation.

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